Dealing with multiple representations in the mathematics classroom: Teachers' views, knowledge, and their noticing

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Abstract

In the last three decades many research studies focused on the topic of multiple representations and their role for learning mathematics. As a result, there is a broad consensus in the scientific community that dealing with multiple representations in the mathematics classroom is a highly relevant matter. However, research addressing the role of the teachers in this context is still scarce. Consequently, this dissertation study raises the question of how much teachers know about and acknowledge this key role of multiple representations for the mathematics classroom. To this end, not only different aspects of teachers' specific professional knowledge and their views were investigated, but also their noticing of changes of representations in instances of student-teacher interaction, which can be seen as a theme-specific noticing. Using a multi-layer model of professional knowledge, this study addresses in particular questions of how such specific aspects of professional knowledge are interrelated and what components of knowledge and views play a role for the teachers' theme-specific noticing.

These research interests were addressed in the scope of three substudies, each of them including two different subsamples (English pre-service teachers/German pre-service teachers, pre-service teachers/in-service teachers, respectively in-service teachers at academic track secondary schools/in-service teachers at secondary schools for lower attaining students), in order to explore the possible roles of cultural background, teaching experience, and school types.

The different aspects of specific professional knowledge and views were assessed by means of a questionnaire instrument. For eliciting the teachers' theme-specific noticing, vignette-based questions were implemented. The data was analyzed mainly by quantitative methods, was however complemented by a qualitative in-depth analysis focusing on how the teachers' theme-specific noticing was informed by different components of their professional knowledge.

The results of this study suggest that the participants did not fully understand the key role of multiple representations for learning mathematics in the sense of their discipline-specific significance and thus indicate specific needs for teacher education and professional development. Differences between the subsamples of teachers became apparent especially regarding the teachers' more situated professional knowledge and their noticing with respect to dealing with multiple representations. Furthermore, the findings of this study underpin the assumption that within the spectrum between teachers' situated and global professional knowledge and views regarding dealing with multiple representations, different components may be distinguished and suggest that in particular all of these components may play a role for teachers' theme-specific noticing.

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Chapter 1

Introduction

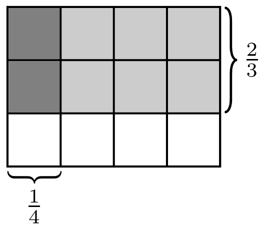
A broad base of research into students' learning with multiple representations in the mathematics classroom has led to a better understanding of the difficulties that many students have with understanding mathematics by emphasizing the crucial role of multiple representations and making connections between such representations (e.g., Duval, 2006; Renkl, Berthold, Große, & Schwonke, 2013). However, teachers' views about how to deal with multiple representations in the mathematics classroom diverge widely: Figure 1.1 presents two statements by German secondary mathematics teachers (translation by the author) that were elicited by the same vignette which is shown in the Figure as well. In the light of empirical findings providing evidence for the influences of teachers' views on their teaching practice (e.g., Staub & Stern, 2002), there is good reason to assume that such contrary views will impact the way these teachers deal with representations in their mathematics classrooms. Consequently, mathematics teachers' views on the role of multiple representations for learning mathematics merit close attention. Corresponding research is however still scarce.

As teachers' views on dealing with multiple representations in the mathematics classroom may be interrelated with what they know about mathematical representations and their role for students' learning, such views should be investigated together with the teachers' specific knowledge.

Bearing in mind that it depends delicately on the context, whether a certain change of representations is rather aid or obstacle for students' understanding (see Acevedo Nistal, van Dooren, Clarebout, Elen, & Verschaffel, 2009), not only views and knowledge about dealing with multiple representations in general, but in particularly more situated, context-specific views are worth focusing on. Consider, for instance, the classroom situation described by the transcript in Figure 1.1 There may certainly be situations in which it is reasonable for a teacher to change from the rectangle representation to the pizza representation for showing how to add fractions – for example, when a student has difficulties identifying the unit and holding it constant. However, in the specific situation outlined above, the student's question refers to the rectangle representation and hence an explanation using another representation without making connections to the original one may lead to difficulties in understanding. For investigating rather situated teachers' views on dealing with multiple representations in the mathematics classroom a situated context has to be provided, for instance by vignettes or specific tasks, which can elicit such views (e.g., Kuntze, 2012).

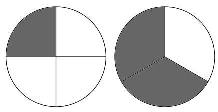
However, eliciting a teachers' views on dealing with representations by means of a vignette presenting a specific classroom situation (or a real classroom situation) requires the teacher to pay attention to corresponding aspects of the specific situation. In the example above, for evaluating whether the change of representations by the teacher is helpful for the student's understanding, it is obviously necessary to pay

T illustrates the calculation $\frac{1}{4} \times \frac{2}{3}$ on the board:

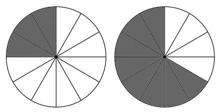


S: And how can you see here what $\frac{1}{4} + \frac{2}{3}$ is?

T: Well, this cannot be seen very well in this picture. For this it would be better to look at pizzas [draws]:



Before we can add the fractions, we have to make all the pieces the same size. Therefore we have to subdivide the pizzas:



Now we see that we have $\frac{3}{12}$ and $\frac{8}{12}$. So, if we add, we get $\frac{3+8}{12} = \frac{11}{12}$.

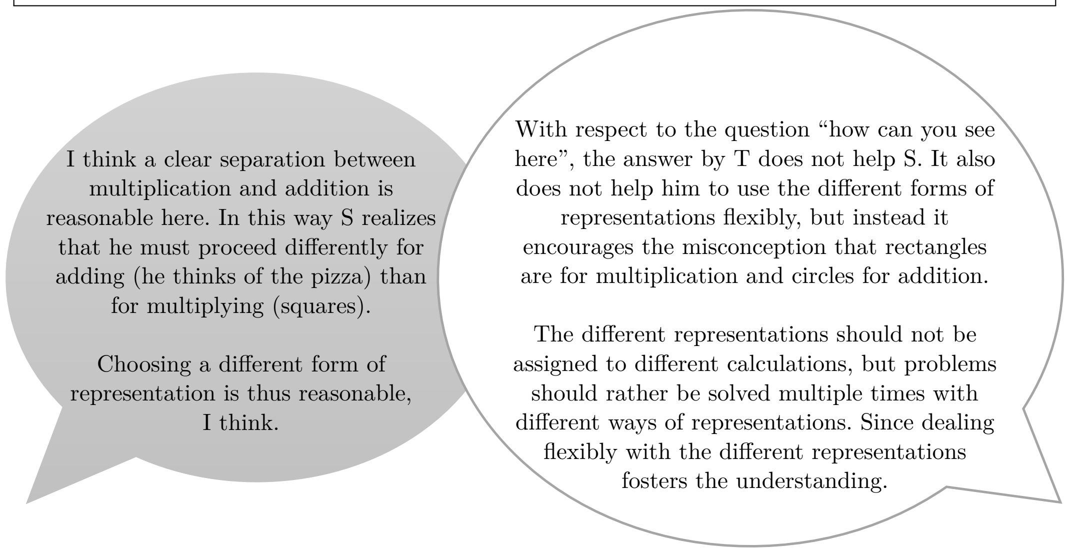


Figure 1.1: Different mathematics teachers' views on how to deal with multiple representations

attention to this change, to notice it. That is, situated contexts require extracting what is relevant in the specific classroom situation or the specific task against the background of corresponding professional knowledge in the sense of noticing (van Es & Sherin, 2002). Hence, in particular the teachers' situated views on dealing with multiple representations in the mathematics classroom are likely to be intertwined not only with their corresponding knowledge, but also with their noticing.

Consequently, the whole complex of specific views, knowledge, and teacher noticing regarding the idea of using multiple representations in the mathematics classroom merits attention. Then again, within the scope of this work it is certainly not possible to provide an extensive investigation of all facets of this complex. Therefore, this work's objective is to spotlight several important facets and also to explore their interrelations. In this way a first, rough sketch can be drawn, which may serve as a starting point for further research which can then investigate particular aspects in greater depth, add further facets or aim at an even broader perspective.

The process of making a reasonable selection regarding the focus of the present study started by choosing a content-domain for setting the content- and situation-specific aspects. As the domain of fractions is particularly well-studied regarding the use of different representations and the significance of multiple representations for the construction processes of students' conceptual understanding (e.g., Ball, 1993a; Padberg, 2009) this domain was chosen for the study of corresponding aspects of teachers' views, knowledge, and noticing.

The following second chapter gives insight into the theoretical background of this study by defining its key terms as well as reviewing literature and research which is important for its design and implications. On this basis, the research questions which are central to this work are derived in the third chapter. Subsequently the design of the study is described, including its methodology and samples. As the study comprises of three substudies which take different perspectives and focus on different subsamples, these three substudies are presented in the fifth chapter, which also outlines how they complement each other with respect to finding answers to the overall research questions. Within this fifth chapter, the three substudies are presented in the form of separate articles, which means that each of them is embedded in an individual, more specific theoretical framework. The results of the substudies, which are at first presented separately within the corresponding articles, are then summarized and discussed from the perspective of the overall research interest in the sixth chapter. Moreover, this last chapter points out theoretical and practical implications of the results of this study and makes suggestions for further research into teachers' views, knowledge, and their noticing with respect to dealing with multiple representations in the mathematics classroom.

Chapter 2

Theoretical Background

The first section of this chapter focuses on the essential role that representations play in the discipline of mathematics. Against this background, in the second section, consequences for learning mathematics are deduced and underpinned by corresponding empirical findings on students' learning with multiple representations of mathematical objects. As this dissertation study focuses in its domain-specific parts on fractions, the role of multiple representations for the students' conceptual understanding of fractions is looked at in more detail in the third section. Subsequently, the focus is moved from the students to the teachers: First, the notions of mathematics teachers' noticing and teachers' professional knowledge are introduced and briefly discussed in general, before they are viewed from the perspective of dealing with multiple representations in the mathematics classroom. Thereby, important aspects of specific views, knowledge, and teacher noticing are spotlighted and possible interrelations are discussed. The last section motivates the investigation of different groups of teachers by focusing on potential differences regarding these aspects of their knowledge, views, and noticing. The distinction between the groups of teachers refers to the country they come from, their teaching experience as being pre-service or in-service teachers, and the type of school they teach at.

2.1 The role of representations in mathematics

Mathematics is the art of giving the same name to different things.
Henri Poincaré (1854 – 1912)

Poincaré (1908/1914) explained this famous quote of his in the following way:

> When language has been well chosen, one is astonished to find that all demonstrations made for a known object apply immediately to many new objects: nothing requires to be changed, not even the terms, since the names have become the same. (p. 34)

Reviewing his work further shows that he had in mind structures such as groups that can be found in many different fields of mathematics and beyond, and that have played an important role for mathematical achievements in the past two centuries. In the words of Poincaré's quote, the "name" group can be given to superficially different "things", since they share an underlying structure. Indeed, it is easily observed that the effectiveness of mathematics often lies in unmasking seemingly different things as "being the same". In particular this is an important reason for why mathematics can often solve problems emerging in other disciplines. Sturmfels (2009), for instance, pointed out that statisticians, computer scientists, physicists, engineers, and biologists have used many different terms in the literature for a certain mathematical object (a specific prime ideal) and that "the concise language of commutative algebra and algebraic geometry can be an effective channel of communication for the[se] different communities" (p. 352).

One may ask however, whether these different things that we can give the same name to are really "different objects" or whether they can rather be seen as different designations for the same object. The latter describes the perspective of Gottlob Frege (1892/1960): In his well-known article "Über Sinn und Bedeutung" [On sense and reference], he gave the following example:

> Let a, b, c be the lines connecting the vertices of a triangle with the midpoints of the opposite sides. The point of intersection of a and b is then the same as the point of intersection of b and c. So we have different designations for the same point, and these names ('point of intersection of a and b', 'point of intersection of b and c') likewise indicate the mode of presentation; and hence the statement contains actual knowledge. (p. 57)

According to Frege (1892/1960), a designation or a name for a mathematical object can take the form of symbol(s) or word(s). Disagreeing with the formalist view that these designations may be equated with the mathematical objects they represent, Frege (1895/1970) has emphasized that designation and object must be distinguished carefully:

> So it is necessary not to overrate words and symbols [...] by mistaking them for the actual things of which they are at most the (more or less accurate) representations. [...] When I write down $1 + 2 = 3$ then I am putting forward a proposition about the numbers 1, 2 and 3, but it is not those symbols that I am talking about. (p. 482)

In this quote Frege pointed out that designations for mathematical objects can be seen as representations. In the present dissertation study the notion representation is understood as "something that stands for something else" (Duval 2006, p. 103). Since, however, not only symbols and words can stand for mathematical objects, representations for mathematical objects can also take different forms, such as graphics or diagrams. Nevertheless, the designations that Frege (1892/1960) referred to in his work are understood as representations for mathematical objects. Figure 2.1 shows an example of some representations for the rational number $\frac{3}{4}$.

Frege (1892/1960) argued that mathematical objects are neither palpable nor directly perceivable, but have to be represented. Taking into account the gap between a designation of a mathematical object and the object itself, Frege (1892/1960, p. 57) came up with his famous distinction between the *sense* (Sinn) and the *reference* (Bedeutung) of a designation: The reference is the object the designation refers to and the concept of sense accounts for the mode of presentation, hence refers to the particular aspect of the mathematical object which is revealed by the designation (Radford, 2002). In Frege's example mentioned above, the designations "point of intersection of a and b" and "point of intersection of b and c" have the same reference, that is, they refer to the same object, namely a specific point in the Euclidean plane, but their sense is different, since they show different aspects of this object.

One of Frege's (1892/1960) remarkable insights concerns the role that different designations of mathematical objects play for the processes of knowledge acquisition in

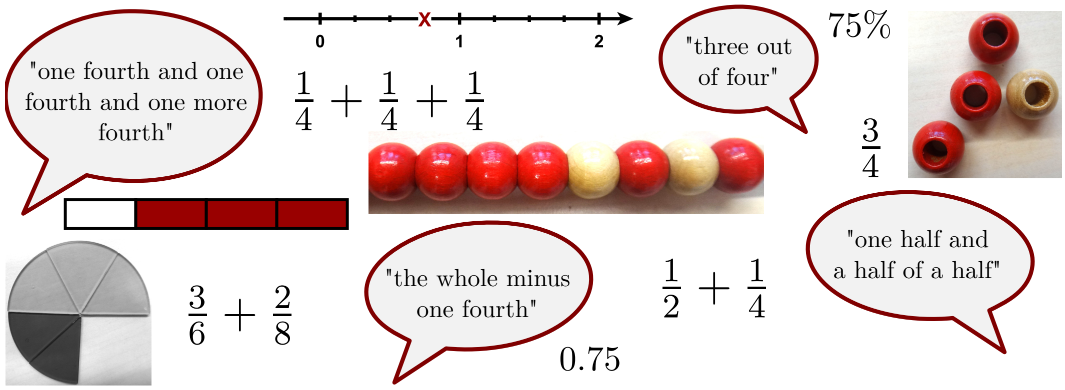


Figure 2.1: Some representations for a rational number

the discipline: He observed that becoming aware of links between different senses of the same object is what discloses knowledge (Radford, 2002). This Fregean perspective can thus be condensed by modifying the above-quoted citation of Poincaré (see Schnotz, 2014a):

Mathematics is the art of giving different names to the same thing.

In this case, the "different names" are different designations or representations and the "same thing" is the mathematical object they refer to. "Giving different names to the same thing" can thus be understood as making connections between different representations of the same mathematical object – which plays an essential role for mathematical understanding as it will become clear in the following sections.

Inspired by Frege's ideas, Raymond Duval (1993; 2006) developed a framework for analyzing the processes of mathematical activity, which he then used for identifying reasons for the difficulties that many students have with comprehension of mathematics, as it will be explained in the following section. Like Frege, Duval pointed out that mathematical objects cannot be perceived directly, but accessing them is bound to the use of representations. Duval focused however not on single representations, but on systems of representations which have rules for performing transformations of representations within the system without changing the mathematical object that is represented. He referred to these systems of representations by the notion of representation *registers* (Duval, 2006, p. 111). The reason why Duval emphasized the possibility of transforming representations of the same object into each other is his observation that transformations of representations are at the heart of mathematical activity (Duval, 2006, p. 107). Building on Frege's insight, Duval (2006) pointed out that mathematical activities always involves substituting some representation for another and that disclosure of mathematical knowledge requires to deal with and to link different senses of a mathematical object. Analyzing the predominant role of transformations of representations regarding mathematical activities more closely, Duval was led to distinguish two types of transformations: *treatments* and *conversions* (Duval, 2006, p. 111). While the notion treatment is used for a transformation of representations within a register of representations, the term conversion refers to a transformation from one register to another. Consider for instance the representations shown in Figure 2.1: Transforming the representation $\frac{1}{4} + \frac{1}{4} + \frac{1}{4}$ into the representation $\frac{1}{2} + \frac{1}{4}$ would be a treatment, whereas the transformation from the number line representation to the strip representation of the fraction would be a conversion, since these representations belong to different registers even though both of them are pictorial. Duval (2006) observed that these two types of transformations have different meanings for mathematical processing. While treatments are often in the foreground of mathematical activities and hence appear to be most important, in many cases it is the conversions of representations which are crucial to gain insight and to facilitate problem solving as well as mathematical argumentation. The reason for this is that the treatments which can be carried out are usually very specific to the register that is used. Thus "passing from one register to another changes not only the means of treatment, but also the properties [of the mathematical object] that can be made explicit" (Duval, 2006, p. 114). Consequently, it is quite often the case that for proving a statement or for solving a problem it is necessary to carry out a conversion into another register, where some insight can be gained (by means of treatments) which can subsequently be carried to the original register by a conversion.

Although Duval's framework appears to be influenced mainly by philosophical ideas, similar insights can be found in psychological research on problem solving, where representations are also considered to play a pivotal role. For the Gestalt psychologists (e.g., Kohler, 1947; Wertheimer, 1959) insight in solving a problem occurs when the problem is restructured, that is, transformed into a suitable representational form, so that the solution becomes suddenly obvious. For the psychologists focusing on information processing (e.g., Newell & Simon, 1972), however, not finding a convenient representation is paramount for successful problem solving, but finding a way to get from an initial state to a goal state in a so-called problem space (see Schnotz, 2014b). Ohlson's (1992) representational change theory constitutes a synthesis of these two perspectives: He observed that problem solving usually requires transforming the problem into a suitable representational form where the initial state can be easily transferred into the goal state (Schnotz, 2014b). This insight of Ohlson's (1992) can be expressed very well using Duval's theory of registers: For solving (mathematical) problems it is often necessary to conduct conversions of representations in order to facilitate expedient treatments. While the Gestalt psychologists focused mainly on the crucial role of conversions for problem solving, the information-processing perspective emphasizes primarily the significance of treatments (see Schnotz, 2014b).

This can be seen as an example for how powerful Duval's theory of registers is for explaining how mathematics and in particular mathematical problem solving work. Moreover, it has been proven very useful for identifying reasons for the difficulties that a lot of students have with understanding mathematics, as it will be discussed in the following section. For these reasons this work uses mainly his framework as basis for developing ideas about what aspects of teachers' knowledge, views, and their noticing regarding dealing with multiple representations merit particular attention. It should however be noted at this point that of course also other authors have developed theories regarding the role of representations in mathematics and in mathematics education. One famous example is for instance Jerome Bruner's (1966) work which proposed three modes of representation: enactive representation (action-based), iconic representation (image-based), and symbolic representation (language-based). Like Bruner, many researchers have focused on classifying representations according to different dimensions such as the sensory channel, the modality, the level of abstraction, the specificity or the type of the representation (for an overview, see Ainsworth, 2006). Such distinctions are often used to explain why one kind of conversion is more difficult for learners to conduct than another.

Duval's theory may in particular be complemented by theoretical frameworks taking the perspective of social constructivism on dealing with mathematical representations (e.g., Cobb, 2002; Font, Godino, & Gallardo, 2013; Meira, 2002): They can give further insight into the role that representations play in the mathematics classroom, since such frameworks emphasize the role of the subject(s) interacting with the representations for the construction processes of their meaning. Meira (1998) pointed out for instance that a representation does not stand for a mathematical object in any obvious, self-explanatory way, but this connection is subject to interpretation and negotiation processes. Similarly, according to Steinbring (2000) creating a connection between representation and object depends on the interaction between the participants in a learning environment. The so-called onto-semiotic approach (Font et al., 2013) argues moreover that even mathematical objects only emerge from the actions and discourse through which they are expressed and communicated.

All of these discipline-specific characteristics of the role of representations for doing mathematics that have been described so far have implications for learners of mathematics. The following section will focus on such emerging consequences for learning mathematics referring in particular to Duval's (2006) theoretical framework.

2.2 Consequences for learning mathematics

Duval (2006) did not only develop a theoretical framework regarding the role of representations in mathematical activity, but he also analyzed problems of comprehension in learning mathematics from the perspective of the framework:

> Here we get to the root of trouble in mathematics learning: the ability to understand and to do by oneself any change of representation register. The troubles that many students have with mathematical thinking lie in the mathematical specificity and the cognitive complexity of conversion and changing representations. (p. 122)

According to Duval (2006) the main reason for the cognitive complexity of conversions of representations is the following problem: On the one hand conversions of representations require to dissociate the represented mathematical object from its particular representation through which it was first used and introduced in the classroom. However, on the other hand, there is a cognitive impossibility of dissociating any mathematical object from its representation, since there is no access to the object apart from representation. He explained further that this conflict often leads to the misconception of learners that two representations of the same object are considered to be two different mathematical objects. Consequently, for these learners the registers of representation remain compartmentalized, disabling them from changing flexibly between different representations. The phenomenon of "compartmentalization" – namely the lack of ability to make connections between different representation registers – has been described and investigated by several researchers in the field (e.g., Gagatsis, Elia, & Mousoulides, 2006; Michaelidou, Gagatsis, & Pitta Pantazi, 2004; Yang & Huang, 2004). Gagatsis, Kyriakides, and Panaoura (2004) examined for instance the role of the number line representation for the ability of second graders to add and subtract natural numbers and for most learners they found a compartmentalization between their ability to carry out the tasks in the symbolic form and their ability to perform the same tasks using the number line representation.

However, since different representations usually emphasize different properties and aspects of the corresponding mathematical object, several distinct representations have to be integrated by learners in order to develop an appropriate concept image (Ainsworth, Bibby, & Wood, 1998; Even, 1990; Janvier, 1987; Tall, 1988; Wittmann, 1981). Consequently, the ability to link different representations and to change flexibly between them has often been seen as an indicator for conceptual mathematical understanding (e.g., Stern, 2001; vom Hofe & Jordan, 2009; Zbiek, Heid, & Blume, 2007). In her article on research in the field of intelligence, Stern (2001, p. 184) even claimed that mathematical competence can be understood as the ability to transform different forms of representation into each other. Students' abilities in dealing flexibly with multiple representations have thus been investigated by several researchers, who have used notions such as "representational fluency", "representational flexibility, "representational versatility" or "representational competence" to describe the underlying construct and to emphasize slightly different aspects (Acevedo Nistal, van Dooren, Clareboot, Elen, & Verschaffel, 2009; Graham, Pfannkuch, & Thomas, 2009; Zbiek et al., 2007). Furthermore, many empirical studies (e.g., Hughes, 1986; Huinker, 1993; Gerster & Schulz, 2000; Moser-Opitz, 2007) have operationalized such constructs as an indicator of mathematical understanding. Stern (2002) emphasized moreover that mathematical knowledge of experts and novices is not distinguished by different degrees of abstraction, but instead the difference lies in its interconnectedness: Accordingly, experts can more easily make connections between symbolic representation registers and corresponding content-related representation registers such as real world situations.

As it was pointed out in the previous section, making connections between different representations and conducting suitable conversions of representations often play an important role for problem solving. It appears hence obvious that mathematical problem solving is facilitated by representational flexibility (e.g., Acevedo Nistal et al., 2014; Ainsworth, 2006; Gagatsis & Elia, 2004; Lesh, Post, & Behr, 1987; Verschaffel, de Corte, & de Jong, 2010). In this context, Lesh and colleagues (1987) stated that "good problem solvers tend to be sufficiently flexible in their use of relevant representational systems that they instinctively switch to the most convenient representation to emphasize at any given point in the solution process (p. 38). Acevedo Nistal and colleagues (2009) dealt with the question as to what characterizes the "most convenient" representation in the problem solving process and they have found that the answer does not only depend on properties of the task at hand, but also on characteristics of the learner and the context. Subsequently, this research group conducted an intervention study regarding the domain of linear functions showing that it is possible to improve students' ability to make flexible representational choices in problem solving in the sense of taking into account such variables (Acevedo Nistal et al., 2014). Improving students' abilities in flexibly using multiple representation is particularly significant in the light of the fact that intelligence is not sufficient for such flexibility, but instead also intelligent learners have to learn it (Stern, 2001).

Consequently, it appears highly relevant to foster students' abilities in making connections and conversions between different representations and in dealing flexibly with such multiple representations in the mathematics classroom (e.g., Graham et al., 2009; Schnotz, 2014b). Accordingly, the national standards in many countries point out the significance of dealing with multiple representations for learning mathematics (e.g., KMK, 2003; NCTM, 2000; Qualifications and Curriculaum Authority, 2007). For example, in the German standards for the mathematics classroom "using mathematical representations" is referred to as one out of six general aspects of mathematical competence. It encompasses "applying, interpreting, and distinguishing different representations for mathematical objects and situations", "recognizing connections between representations" as well as "choosing different representations depending on the situation and purpose and changing between them" (KMK 2003, p 8, translation by the authors). The American standards emphasize among other aspects subsumed under the topic "representation" in particular that all students should be enabled to "select, apply, and translate among mathematical representations to solve problems" (NCTM, 2000, p. 67).

However, using multiple representations in the mathematics classroom is per se no panacea for fostering learners' mathematical understanding and problem solving. The highly demanding cognitive processes pointed out by Duval (2006) lead to negative side effects of learning with multiple representations. The complex cognitive demands associated with using representations start already when considering a single representation, since this representation does usually not stand for any mathematical object in an obvious, self-explanatory way (e.g., Ainsworth, 2006; Font et al., 2013; Meira, 1998). As it was pointed out at the end of the previous section, the connection between representation and represented object is subject to interpretation as well as negotiation processes and has to be established among the participants of the mathematics community or in a learning environment (e.g., Gravemeijer, Lehrer, van Oers, & Verschaffel, 2002; Steinbring, 2000). Therefore, every time when students are faced with a new representation, they have to learn how it is used and interpreted – in the discipline of mathematics or at least in their mathematics classroom. These requirements for learners add to the cognitive demands associated with making necessary connections and conversions between representation registers, which were pointed out at the beginning of this section.

Due to these demanding cognitive processes learning with multiple representations carries the risk of creating obstacles to understanding for students (Ainsworth, 2006; Duval, 2006; Sfard, 2000). There is a broad base of empirical evidence for this phenomenon which shows that learners have difficulties in dealing with multiple representations and fail sometimes to benefit from them (e.g., Ainsworth, Bibby, & Wood, 2002). Ainsworth and colleagues (2002) for instance conducted a study comparing students' learning after they were taught in computational estimation using either formal representations, pictorial representations or a combination of both. Although both of the single representations were found to support learning, the combination had even detrimental effects. In particular students with low prior knowledge in the specific content domain often have problems with the integration of multiple representations (Yerushalmy, 1991). It was observed that they frequently concentrated on a single representation, in many cases the more familiar or concrete one (Cox & Brna, 1995; Scanlon, 1998; Tabachnek & Simon, 1998).

In summary, it can therefore be said that multiple representations play a double role for learning mathematics: On the one hand conversions of representations are essential for mathematical understanding, but on the other hand such conversions can involve excessive demands and thus hinder learning (e.g., Ainsworth, 2006). In other words, the benefits of learning with multiple representations come at the cost of highly demanding cognitive processes. Therefore, learners need to be supported in making connections between different representations in order to benefit from them (Bodemer & Faust, 2006; Rau, Aleven, & Rummel, 2009; Renkl et al., 2013). Several empirical studies have explored different support measures: On the one hand such measures encompass techniques which make salient the elements in the different representations corresponding to each other, such as the use of an integrated format or color coding (e.g., Kalyuga, Chandler, & Sweller, 1999; Renkl et al., 2013). On the other hand instructional procedures encouraging metacognition were implemented, which either use self-explanation prompts or inform the learners about the function of multiple representations (Renkl et al., 2013). Although both kinds of measures were found to help students to benefit from learning with multiple representations, Renkl et al. (2013) argued that the first kind of measures can merely foster making connections between multiple representations on a surface level, but they do not support the integration of these representations in the sense of developing an appropriate multi-faceted concept image.

2.3 Multiple representations in the domain of fractions

As it was emphasized in the previous sections, the significant epistemological role of multiple representations is inherent to the discipline of mathematics and thus applies to all kinds of mathematical contents. The overarching idea of using multiple representations in the mathematics classroom (Kuntze, Lerman, Murhphy, Kurz-Milcke, Siller, & Winbourne, 2011) has however content-specific facets as well, since each content domain typically emphasizes different kinds of representations which are central for the understanding of the corresponding concepts (e.g., Acevedo Nistal et al., 2009; Graham et al., 2009; Kuhnke, 2013). Besides the domain of functions, fractions is one of the best studied content domains regarding the use of multiple representations and their significance for the construction processes of students' conceptual understanding (e.g., Ball, 1993a; Charalambous & Pitta-Pantazi, 2009; Deliyianni, Elia, Panaoura, & Gagatsis, 2008; Niemi, 1996; Padberg, 2009). Hence, corresponding research focusing on mathematics teachers can optimally build on and relate to this research. Since moreover, fractions is a content domain that is a relevant topic at different types of schools, it was chosen for the content-specific parts of the study of teachers' views, knowledge, and their noticing about dealing with multiple representations in the mathematics classroom.

In the following brief overview, a content-specific perspective is adopted in order to take another look at the key role of multiple representations for conceptual understanding in the domain of fractions. Among researchers and practitioners in mathematics education, it is widely acknowledged that teaching and learning fractions is one of the most problematic topics in school mathematics (e.g., Charalambous

& Pitta-Pantazi, 2009; Niemi, 1996; Padberg, 2009; Winter, 1999). In the light of the above reasoning it may not be surprising that one of the main reasons which was identified for learners' problems in understanding fractions is the fact that the concept of fractions is very multi-faceted in the sense of requiring the integration of many different representations (Charalambous & Pitta-Pantazi, 2009; Kieren, 1976; Lamon, 2001; Niemi, 1996). Several authors (e.g., Ball, 1993a; Charalambous & Pitta-Pantazi, 2009; Hefendehl-Hebeker & Prediger, 2006; Kieren, 1976; Malle, 2004; Padberg, 2009; Winter, 1999) have presented a number of different so-called core aspects or subconstructs which are encompassed in the notion of fractions. Whereas these lists of aspects differ slightly from each other in different works and some of the aspects overlap with others, some of the main aspects which are often emphasized are the following: part-whole, operator, ratio, quotient, point on the number line, and measure. Different kinds of fraction representations emphasize typically different aspects of the concept (e.g., Lamon, 2001; Wittmann, 2006). Reconsidering the representations of the rational number $\frac{3}{4}$ presented in Figure 2.1, it may be noted for instance that the pie chart representation highlights in particular the part-whole aspect. In the representation with the four pearls however, the whole as the unit is less visible, whereas in turn the ratio aspect is emphasized more than in the pie chart representation. The fact that usually for each representation there are certain aspects of the construct which are visible only to a limited extent or even not at all, leads to limitations in their usability for teaching and learning purposes. Wittmann (2006) pointed out for instance that the exclusive or predominant use of pie chart representations (e.g., pizza) for fractions is not appropriate as it may lead to an impasse in the learning process, since the operator aspect needed for multiplying fractions is not emphasized by such representations. Similarly, Charalambous & Pitta-Pantazi (2007) argued that representations focusing on the part-whole aspect of fractions are necessary, but not sufficient for building up conceptual understanding of fractions. Consequently, the development of an appropriate multi-faceted concept image of fractions requires integrating and connecting multiple representations (e.g., Ball, 1993a; Brenner, Herman, Ho, & Zimmer, 1999; Siegler et al., 2010).

Even on the symbolic-numerical level there are many different representations for the same rational number: Firstly, expanding and reducing can be seen as treatments of representations not changing the mathematical object that is referred to and secondly, there are also the possibilities to represent a rational number in terms of percentage or a decimal. Despite or even because of this great variability which already exists in symbolic numerical representations of fractions, it is particularly important to go beyond merely formal notations for the sake of conceptual understanding (e.g., Hefendehl-Hebeker & Prediger, 2006). Padberg (2009) emphasized for instance that understanding fraction calculations instead of merely manipulating fractions according to error-prone rules requires making connections between different kinds of representations. Several authors have argued that in particular fostering students' abilities to match symbolic-numerical representations with appropriate pictorial (diagrams, sketches, illustrations) and content-related representations such as real world situations plays a key role for sustainable learning of operations with fractions (e.g., Malle 2004; Prediger 2011; Schnotz, 2014b; see also Stern, 2002).

What was stated in the previous section about the importance of flexibility in dealing with multiple representations for problem solving, is in particular true with respect to the domain of fractions (e.g., Deliyianni et al., 2008; Niemi, 1996). Focusing on this specific content domain, Deliyianni and colleagues found that flexibility in multiple representations was important for solving problems. Furthermore, empirical evidence regarding the double role of multiple representations for learning mathematics was provided: Rau and colleagues (2009) for instance conducted a study using so-called intelligent tutoring systems and they have found that students learned more with multiple pictorial representations of fractions than with a single pictorial representation – however only when they were prompted to self-explain how the pictorial representations relate to the symbolic fraction representations.

2.4 Teachers' noticing, their professional knowledge, and views

In view of the findings about the double role of multiple representations as being both aid and obstacle for learning mathematics, consequences for teaching mathematics and in particular for teaching fractions can be deduced. Firstly, learning environments should provide learners with the opportunity to get to know and to integrate multiple representations of a mathematical object in order to develop an appropriate multi-faceted concept image and flexibility in dealing with multiple representations. Secondly, conversions of representations should not be carried out carelessly, but learners should be fostered explicitly to make connections and to reflect on conversions between representations, so that multiple representations do not become an obstacle to learning in the mathematics classroom. Therefore, teachers have to decide situation-specifically in the classroom interaction whether introducing further representations is helpful and worth the effort of sufficiently fostering connection making or if these representational changes would rather hinder students' understanding. Therefore it is important to pay attention to conversions of representations in the classroom interaction, which may be "critical" in the sense of posing an obstacle to learners' understanding. Consequently, teacher noticing appears to play a significant role for dealing with multiple representations in the mathematics classroom. Before taking a closer look at specific teacher noticing regarding students' learning with multiple representations in the next chapter, the following section briefly focuses on the construct of teacher noticing in general.

2.4.1 Teacher noticing

In recent years the construct of teacher noticing has increasingly drawn the attention of researches in mathematics education, since noticing in the sense of paying attention to and interpreting key features of complex classroom interactions is likely to be a prerequisite for teachers' ability to act adaptively in these situations (Berliner, 2001; Kersting, Givvin, Thompson, Santagata, & Stigler, 2012; Schwindt, 2008; Sherin & van Es, 2009). However, different conceptualizations regarding the notion of teacher noticing can be found in the growing body of research (Sherin, Jacobs, & Philipp, 2011). Whereas according to some authors teacher noticing is merely the act of identifying what is noteworthy in a classroom situation (e.g., Star, Lynch, & Perova, 2011), most researchers also include aspects of reflecting such as interpreting and evaluating these noteworthy events (e.g., Goldsmith & Seago, 2011; Sherin, 2007; van Es & Sherin, 2002). Furthermore, there are also conceptualizations of teacher noticing that encompass in addition the act of deciding how to react to the incidents that are focused upon (e.g., Jacobs et al., 2010).

In this work teacher noticing is understood in the way it was specified in the framework by van Es and Sherin (2002; 2010). They emphasized that teacher noticing does not merely consist in attending to what is significant in a classroom interaction, but that it also includes making sense of and reasoning about what is observed by drawing on corresponding professional knowledge. Sherin (2007) described these two main aspects of teacher noticing in terms of "selective attention" and "knowledge-based reasoning" (p. 385). Whereas knowledge plays an obvious role for the second aspect, it may be assumed that also the identification of noteworthy features of classroom situations depends on the teachers' professional knowledge and views (Bromme, 1992; Kersting et al., 2012; Schwindt, 2008; Schoenfeld, 1998; Sherin, 2007). Schoenfeld (1998) for instance argued that teachers' knowledge and beliefs influence what they determine as important to attend to in complex situations. Moreover, in their empirical study with mathematics teachers focusing on the content domain of fractions, Kersting et al. (2012) found a strong relationship between the teachers' noticing elicited by videotaped classroom situations and their corresponding professional knowledge assessed by a scale of the "Mathematical Knowledge for Teaching" instrument (Ball, Thames, & Phelps, 2008). The fact that the findings by Kersting et al. (2012) also indicate that teacher noticing is a significant predictor of student learning gains emphasizes further the relevance of research in teacher noticing.

To sum up, even though the focus of many research studies appears to shift from teachers' professional knowledge to teachers' noticing, one should bear in mind the following:

> Noticing is essential, but it does not suffice by itself. It takes place within the context of teachers' knowledge and orientations; and the decisions that teachers make regarding whether and how to follow up on what they notice are shaped by the teachers' knowledge (more broadly resources) and orientations. (Schoenfeld, 2011, p. 233)

Consequently, the present study focuses not solely on teachers' noticing, but also on their knowledge and views. This allows in particular insight into how noticing is informed and shaped by professional knowledge.

2.4.2 Professional knowledge and views

As there is a broad consensus in the field of education that what teachers know should have an effect on what their students learn, teachers' professional knowledge has been in the center of interest of many research projects, especially in the last few decades (for an overview, see Depaepe, Verschaffel, & Kelchtermans, 2013). In the course of such research several studies could provide empirical evidence for significant interrelations of aspects of teachers' knowledge and beliefs on the one hand and student learning outcomes on the other hand (e.g., Fennema & Franke, 1992; Hill, Rowan, & Ball, 2005; Kunter, Baumert, Blum, Klusmann, Krauss, & Neubrand, 2011; Staub & Stern, 2002). The question of how the professional knowledge mathematics teachers need for teaching should be conceptualized has however been discussed controversially to date (Depaepe et al., 2013). In the following, three main issues will be pointed out, which have to be considered for the conceptualization of teachers' professional knowledge. In the light of such considerations the model which is used for the present study will subsequently be presented.

The first issue, which is in the foreground of most discussions, is the categorization of different kinds of knowledge a teacher needs in the spectrum between content matter and pedagogy. It is well-known that the starting point for the community's intensified focus on this issue was the introduction of the concept *pedagogical content knowledge* by Shulman (1986a, 1986b). He criticized the research at the time which he perceived as emphasizing almost exclusively general pedagogical aspects of teaching such as classroom management and he called for more attention to the role of content in the teaching context. Such efforts may be emphasized in particular in view of discipline-specific characteristics such as the special role that multiple representations play for learning mathematics, since corresponding knowledge is not included in general pedagogical knowledge, but is instead bound to the discipline and content-specific, as reasoned above. In this spirit, Shulman (1986b) defined pedagogical content knowledge as:

> [...] the most useful ways of representing and formulating the subject that make it comprehensible to others. [...] Pedagogical content knowledge also includes an understanding of what makes the learning of specific topics easy or difficult: the conceptions and preconceptions that students of different ages and backgrounds bring with them to the learning of those most frequently taught topics and lessons. (p. 9)

Shulman (1986a) identified pedagogical content knowledge (PCK) as one out of four major categories of teachers' professional knowledge. The other three categories he suggested were *subject matter content knowledge*, *curricular knowledge* and *general pedagogical knowledge*. The category of subject matter content knowledge that encompasses knowledge related to mathematical contents (Shulman, 1986b) will in the following be referred to simply as *content knowledge* (CK).

In the research area of mathematics education, particular emphasis has been put on the two categories CK and PCK, and accordingly they are at the heart of most models which were developed in this field following the introduction of Shulman's framework (e.g., Ball et al., 2008; Kunter et al., 2011; see also Depaepe et al., 2013). Among these reconceptualizations that have typically refined these categories into subcategories, the most influential one is probably the "Mathematical Knowledge for Teaching" (MKT) framework by Ball and colleagues (2008). Based on a "job analysis" by means of qualitative analyses of teaching practice, this research group has aimed at conceptualizing "the mathematical knowledge that teachers need to carry out their work as teachers of mathematics" (Ball et al., 2008, p. 4). This approach resulted in a refinement of Shulman's categories CK and PCK: Accordingly CK was divided into the subcategories "common content knowledge", "specialized content knowledge", and "horizon knowledge" and PCK encompasses the subcategories "knowledge of content and students", "knowledge of content and teaching" and "knowledge of curriculum" (e.g., Ball et al., 2008).

Although these frameworks by Shulman (1986a, 1986b) and Ball and colleagues (2008) are clearly very influential in the field of mathematics education, they have been criticized by several scholars (Depaepe et al., 2013). First of all, it was questioned whether the proposed categories can be theoretically and empirically distinguished – in particular regarding the subcategories of the MKT framework: Concerning theoretical distinction, Petrou and Goulding (2011) argued for instance that the definition for "specialized content knowledge" given by Ball et al. (2008, p. 400) "mathematical knowledge not typically needed for purposes other than teaching" is by definition PCK. Problematic with respect to empirical distinction is the fact that factor analyses on studies using the MKT test instrument did not support the existence of distinct categories as suggested by the MKT model (Baumert et al., 2010). In view of these difficulties arising from further refinements into subcategories, this study uses only the core categories PCK and CK.

Two further points of criticism regarding the presented models correspond to the two remaining main issues that should be taken into account for the conceptualization of teachers' professional knowledge. Firstly, it was pointed out the need to broaden the concept of teachers' professional knowledge in order to include beliefs as well (Friederichsen, Van Driel, & Abell, 2011; Kuntze, 2012; Petrou & Goulding, 2011). This need arises from the difficulty to differentiate between knowledge and beliefs with respect to (mathematics) instruction (e.g., Lerman 2001; Pajares 1992; Pepin 1999). According to Grossman, Wilson, and Shulman (1989), teachers frequently treat what may objectively be rather be seen as their beliefs regarding teaching as knowledge. Furthermore, theoretical distinctions are difficult to make as well, since what is commonly referred to as knowledge depends on current theories (e.g., Lakatos, 1976). Hence, Pepin (1999) argued: "[. . .] what may have been regarded as knowledge at one time, may be judged as beliefs at another time. Or, once-held beliefs may, in time, be accepted as knowledge in the light of supporting evidence and theories" (p. 3).

A pragmatic response to this issue in conceptualizing teachers' professional knowledge is to include beliefs as aspects of professional knowledge into the model, acknowledging that there is a spectrum between knowledge on the one hand and beliefs/views on the other hand (Kuntze, 2012). Consequently, even though both notions, knowledge and views, are used in this work, since some components may be seen as being rather views or rather knowledge, it should be noted that they are not considered to be strictly separable.

Secondly, it was criticized that Shulman saw PCK apparently as knowledge about teaching that can be acquired and applied independently from specific classroom situations (e.g., Bednarz & Proulx, 2009; Mason, 2008; Petrou & Goulding, 2011). These scholars argued that PCK should instead be linked to and situated in the particular teaching context (Depaepe et al., 2013). This point of criticism reflects the problem that aspects of teachers' knowledge and views can on the one hand be situated and organized episodically, tightly linked to specific classroom situations (e.g., Bromme, 1997; Leinhardt & Greeno, 1986), but on the other hand, more global components which are not tight to specific teaching contexts could also be identified empirically (e.g., Staub & Stern, 2002; Grigutsch, Raatz, & Törner, 1998). Findings of research studies focusing on situated as well as on more global facets of teachers' knowledge and views have shown that these aspects are often interrelated, but can as well be in conflict with each other (e.g., Kali, Goodyear, & Markauskaite, 2011; Kuntze, 2012; Kuntze & Zöttl, 2008). Consequently, conceptualizing teachers' professional knowledge, one should take into account that aspects of knowledge and views may differ in the extent to which they are rather global or situated. Hence, Törner (2002) suggested to structure components of teachers' professional knowledge according to their "globality". Based on Törner's (2002) distinction of three different levels of globality, Kuntze (2012) identified one additional, situation-specific level, resulting in the following four levels: Firstly, aspects of mathematics teachers' professional knowledge can be *global* in the sense of not being tight to any particular content, such as cognitive constructivist and direct transmission views of teaching and learning (Staub & Stern, 2002). Secondly, there are *content domain-specific* aspects, for instance beliefs about stochastics or pedagogical content knowledge regarding geometry. Thirdly, certain components of teachers' professional knowledge may be *related to particular content*, such as views on application aspects of division of fractions or views regarding specific tasks (e.g., Biza, Nardi, & Zachariades, 2007; Kuntze & Zöttl, 2008). And fourthly, as mentioned above there are aspects of professional knowledge *related to a specific instructional situation* (Kuntze, 2012). In order to facilitate the understanding of these different levels of globality, they will be focused upon in more detail and under the perspective of dealing with multiple representations in the mathematics classroom in the next section.

Taking into account these considerations regarding the outlined issues in conceptualizing mathematics teachers' professional knowledge, Kuntze (2012) developed a corresponding three-dimensional model which is illustrated in Figure 2.2 and which is used for the purposes of this dissertation study.

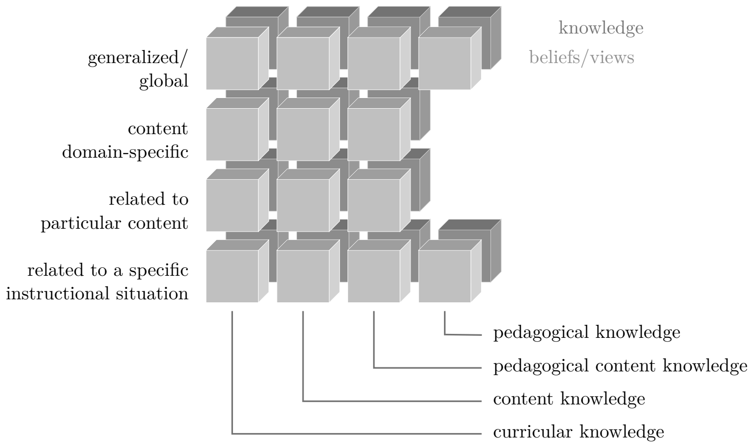


Figure 2.2: Overview model of components of professional knowledge (see Kuntze, 2012, p. 275)

Each of its three dimensions takes into account one of the issues: The spectrum between knowledge and beliefs forms one of these dimensions. A second dimension reflects the possibility to structure teachers' professional knowledge according to the main categories by Shulman (1986a, 1986b) and the different levels of globality constitute its third dimension. Since pedagogical knowledge is by definition not dependent on contents, the corresponding cells in the model are omitted. Although the cells of the model in Figure 2.2 look clearly separated, it is important to bear in mind that they overlap in fact, since the corresponding components of professional knowledge cannot be strictly distinguished (Kuntze, 2012). Nevertheless, the possibilities provided by the model to distinguish between different components have proven useful for investigating the structure of teachers' professional knowledge (Kuntze, 2012; Kuntze & Zöttl, 2008). In particular, the multi-layer model facilitates exploring the question to what extent the teachers' professional knowledge and views on different levels of globality are interrelated or consistent. Whereas some researchers assume that teachers' beliefs are relatively stable and coherent, constituting a kind of personal theory applied to specific contexts and situations (e.g., Bain & MacNaught, 2006; Krzywacki, 2009), others like Hammer and Elby (2002) argued that "there are good reasons to expect that beliefs about knowledge and learning vary with both domain and context" (p. 3). The latter group of scholars draws in particular on the "knowledge-in-pieces" framework developed by diSessa (e.g., 1993), which was applied for the conceptualization of teachers' professional knowledge by Kali and colleagues (2011). According to this theory, teachers make sense of events in specific contexts by activating different pieces of knowledge (so-called p-primes) not necessarily maintaining coherence among their professional knowledge and views across different contexts (Kali et al., 2011).

Against this background of controversial theories, it is particularly relevant to explore interrelations of components of teachers' professional knowledge on different levels of globality. From a practical point of view, corresponding findings may have implications for instance for designing professional development for teachers: In case global views show very little interrelations with corresponding situated, context-specific views, it appears certainly not sufficient for professional development to address teachers' views on a global level. Instead, it would be highly relevant to emphasize explicitly consequences regarding specific contents and classroom situations.

2.5 Theme-specific noticing, professional knowledge, and views about dealing with with multiple representations in the mathematics classroom

Combining the theoretical considerations and empirical findings laid out in the previous sections, this section outlines the core constructs of the present study.

Considering the consequences for teaching mathematics that were deduced from the double role of multiple representations for learning at the beginning of the previous section, one may find that teachers have to balance two potentially conflicting requirements: On the one hand, learners should be encouraged to integrate multiple representations in order to provide them with the prerequisites for conceptual mathematical understanding and problem solving. On the other hand, students should be protected from the excessive cognitive demands which are frequently involved in conversions of representations and which may hinder learning.

Teaching often has to face contradicting demands (Helsper, 1996; Lampert, 1985), not only in a general pedagogical sense, but also discipline-specifically: Such dilemmas bound to mathematics teaching "arise out of the contradictions inherent in weaving together respect for mathematics with respect for students" (Ball, 1993b, p. 7). Pointing out such dilemmas of teaching mathematics, Ball (1993b) referred to Lampert's (1985) work, who specified the notion "dilemma" as "an argument between opposing tendencies within oneself in which neither side can come out the winner" (p. 182) and reasoned further that "from this perspective, my job would involve maintaining the tension between my own equally important but conflicting aims without choosing between them" (p. 182). In view of this definition of the term, dealing with the potentially conflicting requirements arising from the double role of using multiple representations may also be seen as balancing a discipline-specific dilemma of teaching mathematics. Although the conflict between these requirements may at least partly and situation-specifically be resolved by using instructional measures such as described at the end of section 2.2, balancing the dilemma may still be necessary in the sense of not choosing one of its sides in general, but acknowledging both of them and dealing with them situation-specifically in a reflexive way.

Bearing in mind such arising consequences for teaching mathematics, corresponding aspects of teachers' noticing and their professional knowledge as outlined in the previous section certainly merit attention. Corresponding research may allow in particular to identify specific prerequisites and needs for teacher education and professional development. However, such research is still scarce, since most studies assessing mathematics teachers' professional knowledge about representations have focused merely on the evaluation of representations regarding their benefits and drawbacks (e.g., Ball et al., 2008; Kunter et al., 2011). In view of the above reasoning this does however not capture the full spectrum of what teaching with multiple representations in the mathematics classroom has to deal with. In particular the key role of conversions and making connections between different representations for learners' understanding should not be neglected. There are however only a few studies regarding selected facets of teachers' professional knowledge and noticing about dealing with multiple representations in the sense of balancing the benefits and obstacles for understanding mathematics. One of these examples is a qualitative study by Bossé, Adu-Gyamfi, and Cheetham (2011), which focused on teachers' expectations of students being able to perform different conversions of representations. A further example is a study by Charalambous (2008) investigating inter alia also pre-service teachers' noticing regarding dealing with multiple representations. This study will be described in more detail in the following subsection on theme-specific noticing. In spite of these first approaches towards this direction of inquiry, there is clearly a need for research focusing on teachers' professional knowledge, views, and their noticing through the lens of the double role that multiple representations play for learning mathematics. Consequently, the present study aims at contributing to fill in this gap.

Acknowledging the fact that even from the specific angle of dealing with multiple representations, exploring teachers' professional knowledge, views, and their noticing constitutes a very broad research territory, it deems however necessary to narrow the scope of inquiry. To this end, the domain of fractions was chosen for the content-specific aspects of the study (cf. section 2.3). Moreover, regarding the teachers' professional knowledge and views, the present study focuses on components of the categories PCK and CK. Even if general pedagogical and curricular knowledge may also play a role for dealing with multiple representations, PCK and CK appear particularly relevant from the discipline-specific perspective that is taken in this study (cf. section 2.4.2). In view of considerations as to what facets of teachers' professional knowledge, views, and their noticing may in particular affect their teaching with multiple representations, specific constructs were spotlighted in the scope of this dissertation study. In the following, these facets will be presented in detail.

2.5.1 Theme-specific noticing

Balancing the dilemma of multiple representations being aid and obstacle for learning mathematics, teachers often have to decide situation-specifically in the classroom interaction whether introducing a further representation is insightful for the particular learners or if such resulting conversions of representations would rather hinder their understanding. To make such a decision, it is in the first place necessary for the teacher to recognize these instructional conversions of representations as such. This is however not an easy task, as Gerster and Schulz (2000) pointed out that teachers are usually so familiar with the conversions between different forms of representations that these conversions seem to occur "automatically", which means that they are no longer discerned consciously. They argued further that it is hence often difficult for teachers to recognize the excessive cognitive demands that may be imposed to the learners by certain conversions of representations. It is therefore particularly relevant for mathematics teachers to pay attention to conversions of representations in the student-teacher interaction and to show a certain sensitivity for the associated cognitive demands for learners in order not to let multiple representations become an obstacle to understanding.

Relating these thoughts to the noticing framework by van Es and Sherin (2002; 2010) introduced in section 2.4.1, a particular focus for teachers' noticing was chosen – namely the potentially obstructing demands of conversions of representations for students' understanding. This particular kind of noticing will be referred to in the following as *theme-specific noticing*. It encompasses paying attention to conversions of representations in instances of student-teacher interactions and evaluating these conversions by drawing on corresponding professional knowledge. Such evaluation should in particular include the identification of conversions that are potentially hindering for students' understanding by taking into account corresponding criteria such as the following: Is the conversion necessary or especially insightful from the content point of view? Is it likely that the particular student or group of students benefits from the conversion as regards understanding or insight? Are the learners sufficiently supported in making connections between the different representation registers involved?

While some researchers have explored in a broad sense the range of events that teachers notice (e.g., Star et al., 2011; van Es & Sherin, 2010), others have also narrowed their focus to a certain aspect of noticing, most often to students' mathematical thinking (e.g., Jacobs et al., 2010; see also Sherin et al., 2011). Such a focused approach affords a closer inspection of teachers' noticing based on a well-founded theoretical background and corresponding criteria, instead of a mere consideration of all sorts of things teachers may notice in the mathematics classroom. The possibility to take a closer look applies in particular to the interplay of theme-specific noticing with corresponding aspects of teachers' professional knowledge and views. As it was outlined in section 2.4.1, there are theoretical as well as empirical indications for the assumption that teachers' noticing is interrelated with their knowledge and views: Depending on the aspects of their knowledge and views teachers draw on or activate, they may pay attention to different features or occurrences of a classroom situation and depending on what they focus their attention on, they may in turn draw on different components of their professional knowledge to reflect on these incidents. Although these complex processes of interaction between professional knowledge and noticing cannot be explained in terms of cause and effect within the scope of this study, the inquiry can contribute to a better understanding of what components of teachers' professional knowledge play a role for their noticing. In the first instance, one could suppose that teachers' noticing is essentially intertwined with their situated professional knowledge, since this situation-specific knowledge may be closely tight to the kind of classroom situations that set the frame for their noticing. It should however also be taken into account that teachers may draw on their more global knowledge and views and use them as a lens for their noticing. Hence, for exploring which components of professional knowledge play a role for teachers' noticing, different levels of globality merit attention. Moreover, there is also good reason to assume that not only corresponding PCK, but also specific CK is often a requirement for teacher noticing. In order to notice for instance a situation in the mathematics classroom in which it is appropriate to change representations, CK is needed about different representations for the mathematical object at hand, about their connections and about the facets they emphasize. Exploring teachers' theme-specific noticing, both corresponding PCK and CK should therefore be taken into account for gaining insight into the interplay of noticing and professional knowledge.

There is a study of 20 pre-service teachers by Charalambous (2008), which has already taken a step towards this direction of inquiry: Inter alia, this study investigated the pre-service teachers' specific noticing of whether connections between multiple representations were made in the mathematics classroom. Regarding the notion of theme-specific noticing as it was specified above, this kind of noticing relates to an important aspect of theme-specific noticing, namely to one of the criteria for evaluating conversions of representations. Such noticing was elicited by means of a computer-based teaching simulation with cartoon characters set in the content area division of fractions. Furthermore, using the MKT framework and corresponding scales by Hill, Schilling, and Ball (2004), aspects of content knowledge (more specifically "common content knowledge" and "specialized content knowledge"; cf. section 2.4.2) of the pre-service teachers were assessed. Searching for interrelations between the pre-service teachers' specific noticing and these components of their professional knowledge, Charalambous (2008) could however not find any significant correlations. He has thus argued that "further research is needed to better understand under what circumstances teachers' knowledge can inform (preservice) teachers' noticing [...]" (Charalambous, 2008, p. 883).

Considering this call for further research, one may ask however, whether it is indeed the circumstances or rather the components of professional knowledge that require taking a closer examination. Bearing in mind the above reasoning, it appears to be important to take into account different aspects of teachers' professional knowledge and views which may be used as a lens for their (theme-specific) noticing. Moreover, since the ability to notice is often attributed in particular to expert teachers (e.g., Berliner, 1994; Jacobs et al., 2010), in-service teachers may provide a more suitable sample for studying interrelations between their theme-specific noticing and their corresponding professional knowledge and views. Hence, not only pre-service teachers, but also in-service teachers should be investigated concerning this matter.

2.5.2 Views on reasons for using multiple representations

Considering the most general level of globality of pedagogical content knowledge/views (cf. Figure 2.2) from the perspective of dealing with multiple representations, the teachers' views about the role that multiple representations play for students' learning in mathematics appear to be central. In particular the teachers' views on reasons for using multiple representations in the mathematics classroom may play a role for their teaching practice (e.g., Ball, 1993a): Where do teachers see the benefits and purposes of using multiple representations for teaching mathematics? Ball (1993a) reported several studies with American mathematics (pre-service) teachers showing that especially pictorial representations were often seen "primarily as a means to keep and maintain students' attention and interest" (p. 188) and that "making the contexts for learning mathematics fun was a top priority for many" (p. 188). An additional reason for using multiple representations mentioned by these teachers was their potential of supporting learners to remember concepts better. Findings of a study asking German pre-service teachers about purposes of using multiple representations suggest moreover that these pre-service teachers saw in particular the possibility to take into account individual differences of students with respect to their learning type or different preferred input channels (Dreher, 2012a). Although such reasons for using multiple representations for teaching in general may as well be legitimate, in view of the above reasoning about the role that multiple representations play in particular for learning mathematics, the awareness of corresponding discipline-specific reasons appears essential. Seeing the main purpose of multiple representations in making mathematics instruction fun or in supporting students' remembering may justify focusing on representations that are not grounded in meaning instead of making connections between different representations that are important for the development of an appropriate concept image (see Ball, 1993a). Similarly, an overemphasis on taking into account learners' individual preferences may prevent teachers from encouraging students to integrate multiple representations which highlight different aspects of the corresponding object, since these teachers may want to respect the learners' selection of a single, favored representation.

Consequently, such global pedagogical content views of mathematics teachers merit attention as an important component of mathematics teachers' professional knowledge regarding dealing with multiple representation and in particular as a possible lens for the teachers' theme-specific noticing.

2.5.3 Domain-specific views on using multiple representations

Going down one level of globality regarding the teachers' pedagogical content knowledge/views and focusing on the content domain selected for the study, this section addresses domain-specific views on using multiple representations for teaching fractions. Corresponding to global reasons for using multiple representations in the mathematics classroom, there are discipline-specific reasons for teaching with multiple representations of fractions, but there are as well more general arguments such as taking into account individual preferences of learners. Despite the reasoning outlined in section 2.3, which speaks against focusing on a single pictorial register such as pizza representations, a common content domain-specific view is that having a pictorial "standard representation" is desirable/necessary (e.g., Wagner & Wörn, 2012). Reflecting problematic aspects of learning with multiple representations in terms of cognitive demands, teachers may also be concerned that too many different representations of fractions could confuse students. In the light of findings that students' sustainable learning of operations with fractions often depends on making connections with appropriate pictorial and content-related representations (cf. section 2.3), it may moreover be insightful to explore to what extent teachers hold opposing views – such as the worry that highlighting multiple representations of fractions could impede the students' learning of calculation rules for fractions.

From the perspective of the dilemma of multiple representations being aid and obstacle for learning, some of these content domain-specific views reflect rather the side "aid for learning", whereas others emphasize more the side "obstacle for learning". Assessing such views thus affords insight into how the teachers balance this dilemma on the level of domain-specific views on teaching fractions. Moreover, exploring such domain-specific pedagogical content views of teachers may provide a further ingredient helping to understand the interplay of their theme-specific noticing with corresponding professional knowledge.

2.5.4 Evaluating the learning potential of tasks

Taking into account the special role of multiple representations for teaching mathematics may not only concern teachers' noticing in the mathematics classroom, but it may also affect their planning of conceptually rich learning environments. Since task selection and the anticipation of the enactment of these tasks are pivotal for mathematics teachers when planning a lesson (Bromme, 1981), teachers' views on tasks and their learning potential for dealing with multiple representations appear highly relevant for their preparation of corresponding learning environments. Such task-specific views are part of a teachers' pedagogical content knowledge related to particular content (e.g., Biza et al., 2007; Kuntze & Zöttl, 2008). Hence, they are part of the third level of globality with respect to the multi-layer model of mathematics teachers' professional knowledge presented in Figure 2.2. In the light of findings by Baumert et al. (2010) indicating that the learning potential and in particular the potential for cognitive activation of the tasks selected by teachers was a significant predictor of their students' mathematics achievement, it is essential for teachers to evaluate the learning potential of tasks for their mathematics classroom. The learning potential of a task is certainly a broad construct which accounts for its potential to stimulate learners' cognitive activation in the sense of insightful cognitive learning activities (Baumert et al. 2010; Weideneder & Ufer 2013). Hence, the learning potential of a task may be influenced by many factors, such as the way it challenges students' beliefs or its potential to activate prior knowledge (Baumert et al., 2010).

Since tasks are a practical means of encouraging learners to make connections between multiple representations and to integrate them (Duval, 2006), the learning potential of tasks may in particular also depend on the way they use representations. Especially tasks focusing on conversions of representations, which provide insight into their interrelations, have the potential to foster learners' conceptual understanding and their flexibility in dealing with multiple representations. In the following, the learning potential of a task from the perspective of dealing with multiple representations is understood in this sense.

In line with the findings on teachers' global views on reasons for using multiple representations outlined in section 2.5.2, the results of a prior study of pre-service teachers' views on pictorial representations in tasks indicate that many pre-service teachers apparently overemphasized the motivational aspect of pictorial representations (Kuntze & Dreher, 2014). These pre-service teachers hardly acknowledged the learning potential of such pictorial representations which enable students to take an additional approach to mathematical concepts.

In the light of these findings, the question arises as to whether mathematics teachers are preoccupied with the idea of using multiple representations in the sense of "adding a potentially motivating picture" or if they acknowledge the learning potential of tasks focusing on conversions of representations.

2.5.5 Specific CK about multiple representations of fractions

In view of the objective to shed some light on the complex interplay of teachers' theme-noticing and their corresponding professional knowledge, the possible role of specific CK should be taken into account. In particular, there are reasonable grounds to assume that a minimum of specific CK is a necessary (though not sufficient) requirement for successful theme-specific noticing and for an appropriate evaluation of the learning potential of tasks regarding their use of representations: In order to notice conversions of representations in the student-teacher interaction and to evaluate whether they are appropriate for fostering students' conceptual understanding, specific CK is needed about different representations for the mathematical object that is dealt with, about the aspects they highlight and about their mathematical connections. Similarly, evaluating the learning potential of a task with respect to its use of multiple representations, requires a content-specific understanding of the given representations, their interplay and corresponding conversions. In particular, the ability to recognize and reflect connections and conversions between multiple representations and – especially in the domain of fractions – to match symbolic-numerical representations with appropriate pictorial and content-related representations appears therefore to be relevant. Although more general CK may also play a role, it is hence this specific kind of CK which may be especially insightful to assess in view of the aforementioned objectives of the present study. The review of relevant studies in search for a test instrument addressing such specific teachers' content knowledge regarding connections between multiple representations of fractions has shown that there is a need for developing a corresponding instrument: Even though the model for pedagogical content knowledge which was used in the COACTIV study (Kunter et al., 2011) for investigating mathematics teachers' professional knowledge, encompasses "explaining and representing" as one out of three components, as far as CK is concerned, representations appear not to play any explicit role in the research design.

In particular since Charalambous (2008) used scales of the MKT test instrument by Hill an colleagues (2004) for his study of interrelations of pre-service teachers' noticing and their content knowledge, this instrument merits attention in this context. Reviewing the released items (Ball & Hill, 2008) developed by Hill and others (2004) shows that there are items included, which assess CK regarding connections between multiple representations. However, such specific CK was not conceptualized as a separate construct. Hill and colleagues (2004) described the possibility of categorizing their items regarding different kinds of teachers' tasks, where "choosing representations" is one of them. Bearing in mind the considerations above, "choosing representations" does not capture the full spectrum of what is involved in dealing with multiple representations in the mathematics classroom. Furthermore, it may not represent the kind of CK which is most relevant for the teachers' theme-specific noticing. This could be a reason for the fact that in the study by Charalambous (2008) no significant correlations between the pre-service teachers' theme-specific noticing and their CK measured by scales of the MKT test instrument were found. Consequently, specific content knowledge as it was described in this section should be focused upon and a corresponding test instrument should be designed.

2.6 Potential differences between different groups of teachers

Having outlined the aspects of teachers' professional knowledge, views, and noticing, which were addressed by the present study, this section will give reasons for the additional value of taking into account different groups of teachers based on corresponding empirical findings and theoretical considerations. As it will be pointed out in the following, specific knowledge, views, and noticing may in particular depend on cultural settings in different countries, on the stage of a teacher's professionalization and on the type of school where a teacher works.

2.6.1 Different countries

Studies that explore mathematics education and in particular teachers' professional knowledge from different cultural perspectives have typically found inter-cultural differences regarding the constructs focused upon (e.g., An, Kulm, & Wu, 2004; Blömeke & Delaney, 2012; Pepin, 1999). Especially regarding teachers' views, it should be taken into account that some aspects may be culture-dependent (Pepin, 1999). The results of Pepin's qualitative inquiry into mathematics teachers' beliefs with respect to teaching and learning in England, France, and Germany, indicate that such beliefs were influenced by the teachers' cultural environment. Therefore, the investigation of teachers' views should be seen against the background of characteristics and ideas of mathematics teaching in their countries.

Taking into account this possible role of culture, a comparison between English and German pre-service teachers promises to be particularly insightful: Based on her ethnographic study, Kaiser (2002) argued that different educational philosophies in England and Germany have strongly influenced the mathematics classrooms in these two countries. Underpinning this assertion, she presented typical characteristics of mathematics teaching in England and Germany in a contrasting way, which are briefly summarized in the following. According to Kaiser's findings, a central principle of the predominant education philosophy in England is the high priority of the individual. Long phases of individual work are therefore typical for an English mathematics classroom. In this context, English teachers usually appreciate the students' own ways of problem solving as well as their individual notations and formulations. In Germany, however, mathematics classrooms are often organized as class discussions in which ideas are developed collectively. Accordingly, common notations and a precise mathematical language comprehensible by all students is typically seen as being more important. Moreover, Kaiser (2002) pointed out that further distinguishing features of English and German mathematics classrooms are based on contrasting understandings of the role of theory for teaching mathematics. The emphasis on rules, formulae, and arithmetic algorithms, which is often observed in German classrooms, may accordingly be put down to a predominantly scientific understanding of theory in Germany. A rather pragmatic understanding of theory for teaching mathematics in England on the other hand leads to a focus on working with examples and minor relevance of rules and standard algorithms.

Against the background of these findings, comparing English and German pre-service teachers with respect to their views on dealing with multiple representations may provide some insight into which aspects of such views may be rather culture-dependent versus culture-independent. Since pre-service teachers' views on teaching mathematics at the beginning of their teacher education are nurtured from their own experiences as students (e.g., Ball, Lubienski, & Mewborn, 2001; Charalambous, 2008), such views may particularly reflect the characteristics of mathematics teaching in their countries. This approach also affords giving feedback about the research instruments used in this study with respect to their culture-sensitivity and validity.

2.6.2 Pre-service and in-service teachers

Including pre-service teachers as well as in-service teachers into the study allows finding answers to the question as to what difference teaching experience may make for the teachers' theme-specific noticing, their specific professional knowledge and possible interrelations.

There is for instance some evidence indicating that the development of professional knowledge for teaching mathematics is accompanied by a growth in consistency across different levels of globality: Exploring how in-service teachers acquire professional knowledge, Doerr and Lerman (2009) have identified "the teachers' learning as a recurring flow between the procedural and the conceptual" (p. 439), where context-specific responses to problem situations were understood as pedagogical procedures and more general principles for actions as pedagogical concepts. Translated to the terms of the model of professional knowledge used for this study (cf. Figure 2.2), "procedural knowledge" is more situated and less global than "conceptual knowledge". A qualitative study by Turner (2011), investigating beginning primary school teachers concerning the development of their mathematics teaching over four years, may also contribute in this regard: The findings of this study indicate that the growing teaching experience over time enabled the teachers to make connections between knowledge situated in the context of teaching and more global professional knowledge they had been taught at university. Furthermore, Borko and Livingston (1989) accounted for the results of a comparison between expert and novice mathematics teachers by assuming that the so-called cognitive schemata of novices were less interconnected and accessible than those of experts.

Therefore, the question to what degree components of teachers' professional knowledge on different levels of globality are consistent, which was already raised in section 2.4.2 merits attention also from the perspective of its possible role as an indicator for the development of professional knowledge.

Furthermore, the ability to notice is often seen as a characteristic of expert teachers (e.g., Ainley & Luntley, 2007; Berliner, 1994; Jacobs et al., 2010). This suggests in particular that a comparison between pre-service and in-service teachers may yield differences with respect to their theme-specific noticing. Again, interrelations appear to play a significant role, since noticing requires making connections between events that occur in the mathematics classroom and corresponding professional knowledge. Therefore, teachers' expertise may also be reflected in the way how they can link their professional knowledge on different levels of globality and use it as a lens for their noticing. Consequently, expert teachers may also distinguish themselves from novices by stronger interrelations and more consistency between their professional knowledge and their noticing.

Including pre-service as well as in-service teachers, the research presented here may therefore contribute to the identification of such distinguishing features regarding expert and novice teachers and also to the empirical examination of corresponding assumptions within the scope of the study.

2.6.3 Different secondary school types

There are good reasons to assume that the double role of multiple representations for learning mathematics affects in particular learners who are lower-achieving in mathematics (Kuhnke, 2013). Moser Opitz (2009) identified difficulties in performing conversions of representations as a main predictor for low achievement in lower-secondary mathematics. Similarly, Schipper (2005) described such difficulties as one out of four main characteristic of impairments in arithmetic. This suggests that specifically lower-achieving students should be fostered to deal with multiple representations and be supported in making connections between them. Such specific support may be advised as well in the light of the fact that these learners are often particularly affected by excessive demands induced by dealing with multiple representations (e.g., Kuhnke, 2013; Schipper, 2005). Consequently, an even greater sensitivity regarding the double role of multiple representations for conceptual understanding of mathematics is required for teaching lower-achieving students. Hence, one might suppose that mathematics teachers who work with lower-achieving learners are more familiar with the conflicting requirements arising from this double role and more experienced in balancing the corresponding dilemma. In particular one might expect these teachers to be especially aware of the fact that learning with multiple representations can confuse students and pose an obstacle to their understanding.

Against this background, it is expected to be insightful to compare teachers from two different school types regarding their theme-specific noticing and corresponding aspects of their professional knowledge: on the one hand teachers from secondary schools for lower-achieving students (*Hauptschule/Werkrealschule*) and on the other hand teachers at academic-track secondary schools (*Gymnasium*).

Chapter 3

Research Questions

According to the need for research pointed out in the previous two sections, this dissertation study seeks to contribute evidence regarding the following three research interests: The first research interest addresses teachers' views, knowledge, and their noticing regarding the role of multiple representations for learning mathematics. In this context, it is of particular interest to identify corresponding prerequisites and specific needs for teacher education and professional development. Focusing on different aspects of teachers' knowledge, views, and their noticing, this research interest encompasses several research questions. The first three of them concern global, content domain-specific, and task-specific views on using multiple representations (cf. sections 2.5.2, 2.5.3, and 2.5.4). The forth research question focuses on theme-specific noticing (cf. section 2.5.1). And the last research question of the first research interest centers around specific CK (cf. section 2.5.5):

1.1 How much importance do mathematics teachers attach to different (global) reasons for using multiple representations in the mathematics classroom?

1.2 What (content domain-specific) views about using multiple representations for teaching fractions do they have?

1.3 How do they evaluate the learning potential of types of fraction tasks which make use of multiple representations in different ways (conversions of representations vs. unhelpful pictorial representations)?

1.4 Do the teachers' evaluations of specific classroom situations indicate theme-specific noticing, i.e., do they notice conversions of representations and their potentially hindering role for students' understanding?

1.5 What specific CK about dealing with multiple representations in the domain of fractions do they have?

The second research interest focuses on possible interrelations between the aspects of teachers' knowledge, views, and their noticing that are addressed by this study. The first of the corresponding research questions addresses the issues of interrelations between different levels of globality (cf. section 2.4.2). The second research question concerns the possible role of CK (cf. section 2.5.5). The third question regarding this second research interest centers around interrelations between theme-specific noticing and corresponding components of professional knowledge (cf. section 2.5.1). And the fourth research question focuses on interrelations of such aspects of teachers' views, knowledge, and noticing from the perspective of the dilemma of multiple representations being aid and obstacle for learning mathematics (cf. section 2.5).

2.1 To what degree are the teachers' views on those different levels of globality interrelated?

2.2 Is the teachers' CK interrelated with their task-specific views on dealing with multiple representations?

2.3 Is the teachers' theme-specific noticing interrelated with their corresponding knowledge and views? Which components of professional knowledge are used for teachers' theme-specific noticing?

2.4 Is the teachers' awareness of the two sides of the dilemma on the level of views interrelated with their theme-specific evaluations of tasks and their theme-specific noticing?

Finally, the third research interest concerns possible differences between different groups of teachers. In particular this study aims to examine in an exploratory way the role of cultural settings in different countries (cf. section 2.6.1), different stages of teacher professionalization (cf. section 2.6.2) as well as different school types (cf. section 2.6.3) regarding the teachers' specific knowledge, views, and noticing. Namely:

3.1 Does an inter-cultural comparison between English and German pre-service teachers reveal any culture-dependent aspects of their views regarding the role of multiple representations for learning mathematics?

3.2 Do in-service teachers and pre-service teachers differ with respect to their specific knowledge, views, and noticing?

3.3 Do teachers at academic track secondary schools (Gymnasium) differ from their colleagues at secondary schools for lower-achieving students (Haupt-/Werkrealschule) in how they take into account the dilemma of teaching with multiple representations with respect to their views, their evaluations of the tasks, and their theme-specific noticing?

Chapter 4

Design & Methods

The design and methods of the study which was conducted for finding answers to these research questions will be described in the following.

4.1 The questionnaire

For assessing the aspects of teachers' knowledge, views, and noticing focused on by the research questions of the present study, a corresponding paper-pencil questionnaire was designed. An advantage of using a questionnaire instrument is that all of these aspects can be addressed at once in an efficient and standardized way. The questionnaire which was created for this study is based on a previous version which was tested in a pilot study with 145 German pre-service teachers (Kuntze & Dreher, 2014) and subsequently developed further. At the beginning of the questionnaire there were explanations given of the notions representation and pictorial representation in a mathematical context in order to reach a similar understanding of these key terms for the study: "By *representation* we mean the way in which mathematical concepts can be presented and communicated (i.e., algebraically, diagrammatically, descriptively, . . .). In particular, *pictorial representations* are illustrations, diagrams or sketches."

The questionnaire was first designed in German and was then translated into an English version for the English participants of the study. This translation was examined carefully by two native speakers of English, one of whom is also fluent in German and had taught mathematics both in England and in Germany.

Corresponding to the research questions 1.1 to 1.5, the questionnaire encompasses five sections assessing five different facets of teachers' knowledge, views, and their noticing regarding the role of multiple representations for learning mathematics. These questionnaire sections will be described in the following:

4.1.1 Global reasons for using multiple representations

For exploring which reasons for using multiple representations in the mathematics classroom are most important to teachers, the participants were asked to evaluate the significance of possible reasons on a five-point Likert scale (from *not important* to *extremely important*). The four different types of reasons which were given are shown in Table 4.1. These reasons reflect the findings and considerations outlined in section 2.5.2.

Construct (identifier)	Sample item	# items
Necessity for mathematical understanding	Enhancing the ability to change from one representation to another is essential for the development of mathematical understanding.	4
Motivation and interest	They make it easier to keep students' interest.	3
Supporting remembering	Students can use pictorial representations as mnemonics.	3
Learning types and input channels	Different learning types and input channels can be addressed.	3

Table 4.1: Scales regarding reasons for using multiple representations in the mathematics classroom

4.1.2 Content domain-specific views

Against the background of the above reasoning about domain-specific views on using multiple representations for teaching fractions (cf. section 2.5.3) a corresponding multiple choice instrument was designed which focuses on the five constructs presented in Table 4.2. Each construct was measured by means of three multiple choice items. Regarding each of these items the participants could express their approval or disagreement on a four-point Likert scale (from *not true at all* to *completely true*).

Construct (identifier)	Sample item	# items
Multiple representations for understanding	To understand fractions properly, it is necessary to use many different representations in class.	3
Multiple representations for individual preferences	In order to give students the opportunity to choose their preferred type of representation, which they most easily understand, they should be provided with many different representations.	3
One standard representation	It is best to use only one kind of pictorial representation for fractions in lessons, so that you can always come back to this as a 'standard' representation.	3
Fear of confusion by multiple representations	Several different pictorial representations for fractions could confuse students, especially the weaker ones.	3
Multiple representations impede learning rules	If students pay too much attention to pictorial representations, their ability to confidently do calculations with fractions is impeded.	3

Table 4.2: Scales regarding views on dealing with multiple representations for teaching fractions

4.1.3 Task-specific views

In order to examine whether teachers acknowledge the learning potential of tasks focusing on conversions of representations or whether they are rather preoccupied with the idea of using multiple representations in the sense of "adding a potentially motivating picture", the participants were asked to evaluate the learning potential of six fraction tasks. The corresponding questionnaire section was designed such that two types of tasks were contrasted against each other: The three fraction tasks of the first type require carrying out conversions of representations and making connections between different representations. The three tasks of the second type are actually about calculating an addition or a multiplication of fractions on a numerical-symbolical representational level, but they also provide potentially motivating pictorial representations which are however not particularly helpful for the solution, since they cannot illustrate the calculation properly. Figure 4.1 shows samples of tasks of both types. Looking at the example of the type 2 task, it may even be argued that the given pictorial representations are rather confusing than helpful for carrying out the multiplication which is asked for, since neither of the pictures illustrates the operation "taking $\frac{1}{2}$ of $\frac{1}{4}$", but the two fractions merely occur side by side.

Make up a situation or a word problem which is suitable for the calculation $3 \div \frac{1}{4}$ and then use it to solve the calculation.	Do you know what $\frac{1}{2} \times \frac{1}{4}$ is? You can use the pictures below to help: 

Figure 4.1: Samples for tasks of type 1 (left) and of type 2 (right)

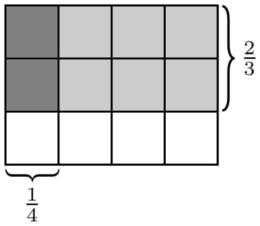
It should be noted that these two types of tasks are clearly not representative for all kinds of fraction tasks and that there are of course fraction tasks using motivating pictorial representations which have a high learning potential. The idea behind this design is however that contrasting these two types of tasks against each other allows insight into whether and where teachers see the learning potential of multiple representations with respect to fraction tasks.

In terms of context information for evaluating the tasks, the participants were informed that all six tasks were created for an exercise about fractions in school year six. The teachers were asked to evaluate the learning potential of these tasks by means of the same three multiple-choice items for each task. A sample item is: "The way in which representations are used in this task aids students' understanding." The participants could express their approval or disagreement regarding these items on a four-point Likert scale (from *not true at all* to *completely true*).

4.1.4 Theme-specific noticing

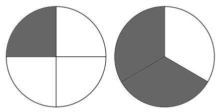
For assessing the teachers' theme-specific noticing a vignette-based format was applied. The vignettes which were given to the participants are transcripts of four fictitious classroom situations. Even though videos would have the advantage of a more

T illustrates the calculation $\frac{1}{4} \times \frac{2}{3}$ on the board:

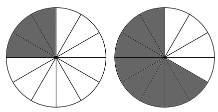


S: And how can you see here what $\frac{1}{4} + \frac{2}{3}$ is?

T: Well, this cannot be seen very well in this picture. For this it would be better to look at pizzas [draws]:



Before we can add the fractions, we have to make all the pieces the same size. Therefore we have to subdivide the pizzas:



Now we see that we have $\frac{3}{12}$ and $\frac{8}{12}$. So, if we add, we get $\frac{3+8}{12} = \frac{11}{12}$.

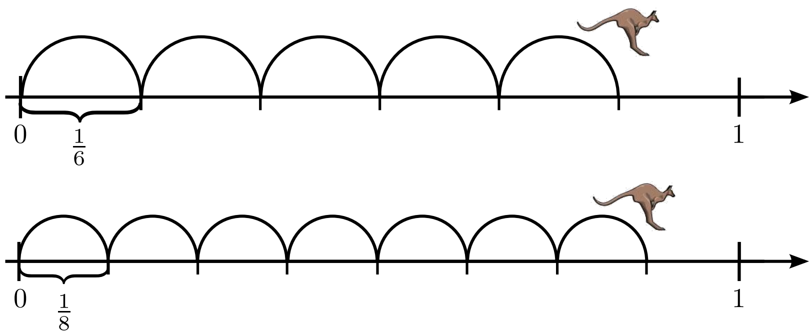
Figure 4.2: Sample vignette # 1 (T: teacher, S: student)

realistic presentation of classroom situations, text vignettes in form of transcripts were used, since they better afford controlling for unintended disturbing factors like for example external characteristics of the teacher. As this was a first attempt to design an instrument to elicit teachers' theme-specific noticing and also to assess it in a quantitative way, it was essential to avoid such potential disturbing factors for the sake of a clear and coherent design. To this end, all four classroom situations have the following characteristics in common: A student makes a comment revealing a misconception or asks a question and thus prompts the teacher to react somehow. The following reactions involve a change of representations, that is, the fictitious teacher uses another representation than the student without making explicit connections between the given and the newly introduced representation. This means that a conversion of representations is conducted by the teacher that is potentially hindering for students' understanding and not necessary from the content point of view. Figure 4.2 and Figure 4.3 show examples of such vignettes.

The first vignette (cf. Figure 4.2) was already presented in the introduction (cf. Figure 1.1) to illustrate how such classroom situations can elicit very distinct views. In this classroom situation a student asks about how one can see the addition of two fractions in the rectangle representation which was previously drawn on the board by the teacher. The teacher, however, uses a pizza representation for explaining the calculation without making explicit connections between these two pictorial representations. It may even be argued that it is here easier to show the addition of the fractions with the rectangle, since the subdivision into twelfths is already

S: The fractions $\frac{5}{6}$ and $\frac{7}{8}$ are of equal size! There is always one piece of the whole missing.

T: Hmm, let's have a look at this on the number line: [draws]



Now you can see that the kangaroo that jumps five times $\frac{1}{6}$ does not get as far as the kangaroo that jumps seven times $\frac{1}{8}$, can't you?

Figure 4.3: Sample vignette # 2 (T: teacher, S: student)

given. Therefore, forcing the student to engage with another representation in this situation may not be deemed necessary or appropriate, even in case previously the pizza representation had been used for illustrating the addition of fractions in this class.

In the classroom situation presented by the transcript in Figure 4.3, a student claims that the fractions $\frac{5}{6}$ and $\frac{7}{8}$ were of equal size, reasoning that "there is always one piece of the whole missing". As a reaction to this misconception the teacher draws two number lines and kangaroos on the board, where one kangaroo jumps five times $\frac{1}{6}$ and the other seven times $\frac{1}{8}$, both starting at zero. He or she argues then that the first kangaroo does not get as far as the second one. With this representation the teacher does, however, not emphasize the key fact that the pieces missing of the whole are not of equal size. Therefore, his or her reaction introduces a new representation which is not appropriately connected with the student's verbal representation when speaking of missing pieces of the whole.

In order to assess the participants' theme-specific noticing, they were asked the following question with respect to these four vignettes: "How much does this response help the student? Please evaluate the use of representations in this situation and give reasons for your answer". It should be noted that with this question the participants were prompted to evaluate the use of representations. Alternatively it would have been possible to ask them for a more general evaluation of the teacher's reaction. A disadvantage could have been however that a lack of time or space to list all possibly interesting aspects of the teacher's reaction may prevent the participants' answers from indicating their theme-specific noticing. Consequently, a more focused question was chosen. A similar methodological approach becomes evident for instance in the question format of the study by Jacobs and colleagues (2010) on teacher noticing of children's mathematical thinking, as well.

For finding answers to the corresponding research question, the teachers' answers were analyzed with a focus on two main aspects: paying attention to the change of representations and sensitivity to potentially negative effects of this change of representations for the student's understanding. Accordingly, each answer was coded

Evaluation	Reference to change of representations	
	No	Yes
None		
Positive	"I find the reaction acceptable and good. Especially for adding fractions, pizzas are still most suitable."	"In this way S realizes that he must proceed differently for adding (he thinks of the pizza) than for multiplying (squares). Choosing a different form of representation is thus reasonable, I think."
Balanced	"For many students this is too difficult, although the addition is well explained in the example."	**"This reaction can help S to answer his question himself, but it will probably rather frustrate him, since there is no response to his question. (...) Jumping to a new representation is rather confusing."**
Negative	"The student is confused, since multiplication suddenly turns into addition."	**"T. could and should have shown the addition using the partitioned rectangle. The change hardly helps the student."**

Table 4.3: Coding illustrated by means of sample answers to the vignette shown in Figure 4.2; bold answers were considered to indicate theme-specific noticing

in a top-down approach regarding to whether it shows that the participant has paid attention to the conversion of representations and whether he or she has seen it critically. Those answers for which both is true (i.e., reference to change of representations and negative or balanced evaluation) were considered to provide an indicator of the theme-specific noticing that this study targets (cf. e.g., bold answers in Table 4.3). Table 4.3 demonstrates this coding by means of sample answers from the data regarding the vignette shown in Figure 4.2. In a preliminary step the answers were coded in a more fine-grained way regarding the question "Which role does the teacher's use of representations play in the justification for this evaluation?" It was distinguished between the following categories:

- no justification given for the evaluation
- justification without referring to representations
- justification referring to representation(s),
  but not to the change of representations
- justification referring to the change of representations

These categories allow for a more detailed analysis with respect to the question to what extent dealing with representations plays a role for the participants' evaluations of the fictitious teachers' reactions. Since for assessing the theme-specific noticing that is targeted by this study, paying attention to the change of representations is however essential, in a second step the first three categories could be merged to a new category, namely "no reference to change of representations", as it was done in Table 4.3.

All answers by the participants were double-coded by the author and a student research assistant with high inter-rater reliability, where Cohen's kappa was always around .9. Discrepancies were resolved through discussion in which an agreement could always be reached. Based on this top-down coding a corresponding score for theme-specific noticing could be assigned to each participant, which counts in how many (out of four) cases his or her answer indicates that the change of representations and its critical role for the student's understanding was noticed.

4.1.5 Specific content knowledge

For investigating to what extent the participants were able to match symbolic-numerical representations of fractions and their operations with appropriate pictorial and content-related representations a test instrument focusing on such specific CK was designed. In Figure 4.4 and Figure 4.5 sample items from this test are presented.

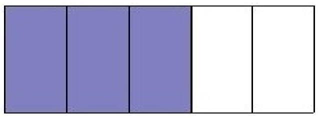
Please change the diagram, if necessary, so that $\frac{3}{5}$ of $\frac{1}{4}$ is shaded. Otherwise just tick the box on the right-hand side.		

Figure 4.4: Sample item # 1 of the CK test

Please change the score, if necessary, so that the home team has scored exactly $\frac{1}{5}$ of the goals. Otherwise just tick the box on the right-hand side.	Standing in a soccer game: home away 1 : 5	

Figure 4.5: Sample item # 2 of the CK test

In total this test instrument encompasses eight such items which have all in common that given (incorrect) conversions between representations had to be checked and corrected or a conversion had to be carried out. All items focused on conversions between a symbolic-numerical representation register on the one hand and a pictorial register (cf. sample item #1) or a content-related register (cf. sample item #2) on the other hand. In a top-down approach, the answers to the tasks were scored dichotomously as being right or wrong.

4.2 The sample

In view of the third research interest, different groups of teachers had to be included in the study. The data from pre-service teachers could be collected at the beginning of university courses. Thanks to a cooperation with Prof. em. Dr. Stephen Lerman at London South Bank University also English pre-service teachers could be recruited for the study. German in-service teachers teaching in Baden-Württemberg were contacted via e-mails to the headmasters of their schools. A meeting was arranged at their school, where they answered the questionnaire.

The pre-service teachers as well as the in-service teachers completed the questionnaire instrument in the presence of the author or a student research assistant and they were given as much time as they needed.

In the following, the characteristics of the subsamples of the study will be described.

English pre-service teachers (primary)

139 English pre-service teachers (99 female, 22 male, 18 without data) took part in the study. They were at the beginning of their first year of teacher education at university preparing to teach at primary level. Since pre-service teachers in England have already finished a university degree before they can start a one-year teacher training, the participants had a mean age of 27.9 years (SD = 6.9). The broad majority of these pre-service teachers had not been studying mathematics since secondary school.

German pre-service teachers (primary)

The subsample of German pre-service teachers preparing to teach at primary level in Baden-Württemberg encompasses 219 participants (183 female, 26 male, 10 without data). Like the English pre-service teachers, they were at the beginning of their first year of teacher education at university. Since German pre-service teachers enter their university studies directly with a 4 to 5 year teacher education program, they were however on average younger than the English participants: They had a mean age of 20.7 (SD = 2.5).

German pre-service teachers (GY)

67 German pre-service teachers (33 female, 34 male) preparing to teach at academic-track secondary schools (Gymnasium) in Baden-Württemberg participated in the study. They had a mean age of 21.4 years (SD = 2.2) and their average semester of teacher education at university was 3.3 (SD = 0.97).

German in-service teachers (GY)

The subsample of German in-service teachers at academic-track secondary schools (Gymnasium) in Baden-Württemberg includes 77 participants (35 female, 39 male, 3 without data). They were on average 40.6 years old (SD = 11.8) and their mean teaching experience for mathematics was 12.4 years (SD = 11.5). This subsample will in the following be referred to as *GY teachers*.

German in-service teachers (HWR)

The second sample of in-service teachers consists of 25 participants (15 female, 10 male) teaching at secondary schools for lower-achieving students (Haupt-/Werkrealschule). These teachers had a mean age of 39.9 years (SD = 11.3) and they had an average teaching experience for mathematics of 10.8 years ($SD = 9.5$). These teachers will in the following be referred to as *HWR teachers*.

4.3 Data analysis

Most of the constructs addressed in this study were measured by means of multi-item scales. This applies in particular to the questionnaire sections focusing on teachers' views. In order to determine whether the theory-based structure of these measures were reflected by the empirical data, confirmatory factor analyses (CFA) were conducted using AMOS 21 software (Arbuckle, 2012). Regarding the questionnaire instrument about task-specific views, for instance, the assumption that the participants' evaluations of the learning potential of two tasks of the same type are more similar than those of two tasks of different types suggest that two second order factors can be empirically separated which represent the evaluations of the evaluations of the types of tasks. The appropriateness of this theory-based model was examined empirically by means of a CFA. Such analyses encompass the calculation of both measures of global and of local fit. Measures of global fit evaluate whether the empirical associations among the manifest variables are appropriately reconstructed by the model (Kline, 2005). Measures of local fit indicate whether each construct can be reliably estimated from its indicators (Bagozzi & Baumgartner, 1994).

Differences between two subsamples according to the third research interest of the present study were addressed by conducting T-tests. In order to measure to what degree different aspects of teachers' knowledge, views, and their noticing addressed by this study are interrelated in a quantitative way (cf. second research interest) Pearson's correlation coefficients were calculated. Moreover, for exploring different profiles of teachers' views as targeted by this inquiry, hierarchical cluster analysis using Ward's method was conducted. Complementing qualitative evaluations did not only consist in double-coding the participants' answers regarding the vignette-based questionnaire section as described above, but also encompass analyses of cases. In particular with respect to research question 2.3, teachers' answers were investigated regarding which components of professional knowledge were used for their theme-specific noticing. This analysis was done by means of a consensus coding approach. The examples of teachers' answers were selected to show the existence of certain phenomena and thus the lack of generalizability inherent to such a case-based design is not problematic.

Chapter 5

The Overall Study: A Puzzle of Three Pieces

For finding answers to the research questions of this study, three substudies were considered about which corresponding articles were written. Each of these substudies focused on another research question from the third research interest, namely on a different comparison of two groups of teachers. Therefore, the participants in the first substudy were English and German pre-service teachers, the second substudy centered around German pre-service and in-service teachers, and the third substudy addressed German in-service teachers from two different school types (GY/HWR). The three substudies differ however not only in the samples they involved, but also in the parts of the questionnaire which they focused on. Accordingly, they addressed different subsets of the research questions of the overall study. Table 5.1 gives an overview of which subsamples, parts of the questionnaire, and research questions were dealt with in the substudies.

Parts of the questionnaire	1. substudy	2. substudy	3. substudy
Global reasons	•	•	
Content domain-specific views	•	•	•
Task-specific views	•		•
Theme-specific noticing		•	•
Specific CK	•	•	
Subsamples			
English pre-service teachers (primary)	•		
German pre-service teachers (primary)	•		
German pre-service teachers (GY)		•	
German in-service teachers (GY)		•	•
German in-service teachers (HWR)			•
Research questions			
1.1	•	•	
1.2	•	•	•
1.3	•		•
1.4		•	•
1.5	•	•	
2.1	•		
2.2	•		
2.3		•	
2.4			•
3.1	•		
3.2		•	
3.3			•

Table 5.1: Overview of the three substudies

Aside from these differences in the substudies, the corresponding three articles also differ in the perspective they take regarding the overall research interest in the sense that each of them has its own characteristic focus: Whereas the first article addresses mainly pedagogical content views on the role of multiple representations in the mathematics classroom, the second article centers around teachers' theme-specific noticing and the third article reviews these aspects from the perspective of the dilemma of multiple representations being aid and obstacle for learning. In the following, there is a brief overview of each of these three articles and the way they contribute to the puzzle of the overall study.

The first article: "Why use multiple representation in the mathematics classroom? Views of English and German pre-service teachers"

In this article views of English and German pre-service teachers on the role of multiple representations for learning mathematics are explored. This affords in particular identifying specific needs and prerequisites for their teacher preparation programs. The article focuses on the pre-service teachers' specific views on different levels of globality and it was investigated to what degree such views were interrelated or consistent. As outlined in section 2.6.1, the comparison between English and German pre-service teachers may also allow some insight into culture-dependent facets of such views.

The second article: "Teachers' professional knowledge and noticing – The case of multiple representations in the mathematics classroom"

Whereas the first article focuses mainly on views, the construct of teachers' theme-specific noticing as it was described in section 2.5.1 lies at the heart of this second article. However, since teachers notice through the lens of their professional knowledge and views, there is also a focus put on interrelations between the teachers' theme-specific noticing on the one hand and their corresponding knowledge and views on the other hand. A quantitative approach was complemented by a qualitative analysis of cases to show how theme-specific noticing can draw on knowledge and views on different levels of globality and to illustrate how theme-specific noticing may fail (i.e., due to lacking specific CK). While the first substudy had its focus on pre-service teachers, this second substudy also included in-service teachers. As it was pointed out in section 2.6.2, noticing is often seen as a characteristic of expert teachers and hence such a comparative design affords insight into potential differences between expert and novice teachers regarding their theme-specific noticing.

The third article: "Teachers facing the dilemma of multiple representations being aid and obstacle for learning – Evaluations of tasks and theme-specific noticing"

This article is characterized by its emphasis on the double role of multiple representations being aid and obstacle for learning mathematics. It revisits some of the constructs that were explored in the first two substudies under the perspective of the corresponding dilemma as it was described in section 2.5. Balancing this dilemma may on the one hand concern the level of views in the sense of generally being aware of both sides. Thus, different profiles of teachers' views on dealing with multiple representations of fractions were explored. On the other hand, however, balancing the dilemma should also become evident in the teachers' reflections related to situated contexts, in particular regarding their evaluations of tasks and their noticing of significant events in classroom situations. In view of the considerations outlined in section 2.6.3, this substudy includes mathematics teachers from different secondary school types (higher-achieving and lower-achieving students), allowing to explore whether these teachers differ in the way they deal with the dilemma associated with using multiple representations.

5.1 Article 1

Why use multiple representations in the mathematics classroom? Views of British and German pre-service teachers

Anika Dreher, Sebastian Kuntze, Stephen Lerman

Abstract
Dealing with multiple representations and their connections plays a key role for learners to build up conceptual knowledge in the mathematics classroom. Hence, professional knowledge and views of mathematics teachers regarding the use of multiple representations certainly merit attention. In particular, investigating such views of pre-service teachers affords identifying corresponding needs for teacher education. However, specific empirical research is scarce. Taking into account the possible role of culture, this study consequently focuses on views about using multiple representations held by more than 100 English and more than 200 German pre-service teachers. The results indicate that there are culture-dependent aspects of pre-service teachers' views, but also that there are common needs for professional development.

Keywords: Multiple representations, Views, Pre-service teachers, Trans-national design, Fractions

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Introduction

There may be many good reasons for using multiple representations for teaching in general, such as for instance the possibility of taking into account the learners' individual differences and preferences. However, since representations play a special role in mathematics, there are also discipline-specific reasons for using multiple representations. As mathematical concepts can only be accessed through representations, they are crucial for the construction processes of the learners' conceptual understanding (Duval, 2006; Goldin & Shteingold, 2001). The awareness of such discipline-specific reasons can clearly influence the teachers' abilities to design rich learning opportunities. For instance, acknowledging that only the combination of different representations affords the development of a rich concept image (Tall, 1988) may better support teachers in designing mathematical activities than seeing the main purpose of multiple representations in keeping pupils' interest. Hence, specific knowledge and views about using multiple representations merit attention – in particular when it comes to professional development. Exploring such views of pre-service teachers at the beginning of their teacher education affords the identification of specific needs and prerequisites. Consequently, this study focuses on pre-service teachers' views on using multiple representations in the mathematics classroom. We use a trans-national design with English and German pre-service teachers to explore whether these views are strongly culture-bound. In line with a multi-layer model of professional knowledge, such views are examined on different levels of globality to find out how consistent general views on using multiple representations are with corresponding content domain- and task-specific views. For the content domain-specific parts of this study we chose the domain of fractions, because of the high relevance of multiple representations specifically in this content domain (e.g., Ball, 1993a; Brenner et al., 1999). Furthermore, possible interrelations with specific content knowledge (CK) are explored. In the following first section, we introduce the theoretical background of this study; the second and third sections present research questions and the research design. Results are reported in the fourth section and discussed in the fifth section.

Theoretical background

The theoretical background of this study includes several aspects which constitute the structure of this section. First, we focus on the special role that representations play for teaching and learning mathematics. The second part is about (pre-service) teachers' views on dealing with multiple representations in the mathematics classroom as aspects of pedagogical content beliefs in the context of a model for teachers' professional knowledge. Particular emphasis is put on reasons why different levels of globality should be taken into account when exploring such views. After giving some answers to the question as to why this study involves pre-service teachers of two different countries, the last part of this section revolves around possible interrelations between the views in the centre of this study and specific CK.

Multiple representations in the mathematics classroom

In mathematics and consequently also in mathematics classrooms representations play a special role. According to Duval (2006) mathematical objects are not directly accessible and hence experts as well as learners have no choice other than using representations when dealing with those objects. We take the notion "representation" to mean something which stands for something else – in this case for an "invisible" mathematical object (Duval, 2006; Goldin & Shteingold, 2001). Figure 5.1 shows an example of some representations for a fraction.

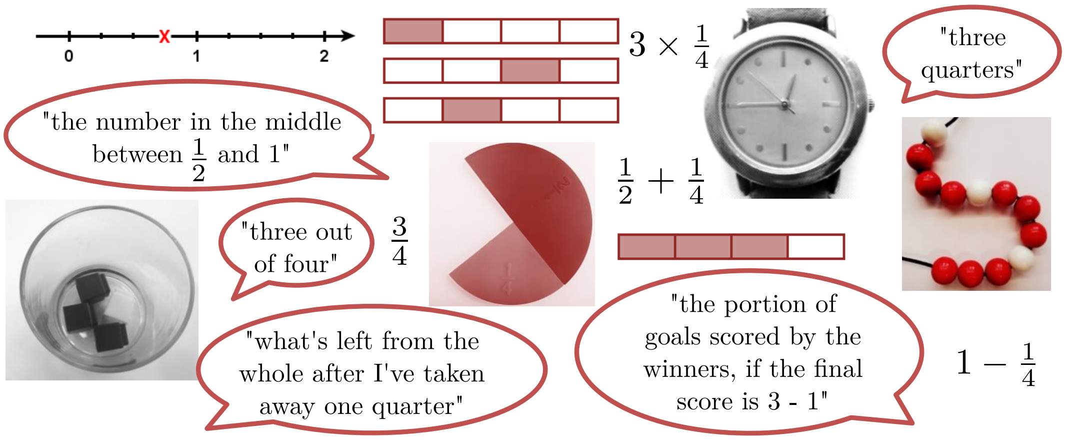


Figure 5.1: Some representations for a fraction

The example illustrates that usually a single representation can only emphasise some properties of a corresponding mathematical object. For instance, the string of pearls emphasises the ratio aspect of the fraction, whereas the pie chart rather shows the fraction as being a part of a whole. Hence, multiple representations which can complement each other are usually needed for the development of an appropriate concept image (Ainsworth, 2006; Elia, Panaoura, Eracleous, & Gagatsis, 2007; Even, 1990; Tall, 1988; Tripathi, 2008). Consequently, it is a sound assumption that using multiple representations is important for developing pupils' mathematical understanding. And indeed, there is substantial empirical evidence for the positive effects of learning with multiple representations on pupils' conceptual understanding (Ainsworth, 2006; Rau et al., 2009; Schnotz & Bannert, 2003). Rau and colleagues (2009) for instance conducted a study with so-called intelligent tutoring systems and found that pupils learned more with multiple pictorial (i.e., graphical) representations of fractions than with a single pictorial representation – but only when prompted to self-explain how the pictorial representations relate to the symbolic fraction representations. The fact that this positive result comes with a certain restriction is not a coincidence: Various studies have shown that providing pupils with multiple representations does not per se foster pupils' learning, since integrating and connecting the different representations is usually difficult for pupils (Ainsworth, 2006; van der Meij & de Jong, 2006). It should also be noted that a representation does not stand for a mathematical object in an obvious way. This connection depends on interpretation and negotiation processes (Gravemeijer et al., 2002; Meira, 1998) and it is usually created in the interaction of the participants in a learning environment (Steinbring, 2000). Therefore, learners need to be supported in constructing meaning with respect to every single mathematical representation and also in making connections between different representations. Findings from several studies underpin this reasoning by showing that pupils need to be encouraged to actively create connections between representations in order to benefit from using multiple representations (Bodemer & Faust, 2006; Renkl et al., 2013). To sum up, fostering the learners' competencies in dealing with multiple representations should be a central goal in the mathematics classroom. Corresponding objectives can be found in many national standards, where dealing with representations is described as an important aspect of mathematical competence (e.g., KMK, 2003; NCTM, 2000; Qualifications and Curriculum Authority, 2007). In the English national curriculum for mathematics "representing" is considered as one of the "essential skills and processes in mathematics that pupils need to learn to make progress" (Qualifications and Curriculum Authority, 2007, p. 142). Even more explicitly, the German national standards characterise "using mathematical representations" as one out of six general mathematical competences, which includes "applying, interpreting, and distinguishing different representations for mathematical objects and situations", "recognising connections between representations" and "choosing different representations depending on the situation and purpose and changing between them" (KMK, 2003, p 8, translation by the authors).

Whereas "using multiple representations" is an overarching idea which is relevant for all parts of mathematics (see Kuntze et al., 2011), there are content domains in which multiple representations are exceedingly significant for pupils' learning. "Fractions" – which is the focus of the domain specific parts of this study – is considered as one of them (e.g., Ball, 1993a; Brenner et al., 1999; Siegler et al., 2010). As different representations can emphasise different core aspects of the concept of fraction (e.g., part-whole, ratio, operator, quotient, etc., see e.g., Charalambous & Pitta-Pantazi, 2007; Malle, 2004; cf. also Figure 5.1), the development of an appropriate multi-faceted concept image of fractions requires integrating and connecting multiple representations. In particular, fostering the pupils' abilities to match symbolic-numerical representations with appropriate pictorial (diagrams, sketches, illustrations) and content-related representations such as real world situations can play an important role for sustainable learning of fraction calculations (e.g., Ball, 1993a; Malle, 2004). Taking the example shown in Figure 5.1, it may for instance support conceptual understanding to establish a connection between the representation $\frac{1}{2} + \frac{1}{4}$ and the pie chart representation where a quarter pie is added to a half of a pie.

(Pre-service) teachers' views on dealing with multiple representations

Since it is well-known that teachers' views about teaching and learning mathematics influence their instructional practice and what their pupils learn (e.g., Kunter et al., 2011; McLeod & McLeod, 2002), it is also likely that in particular their views about dealing with multiple representations play an important role. For exploring such views this study uses the model of teacher professional knowledge which is shown in Figure 5.2 (see Kuntze, 2012).

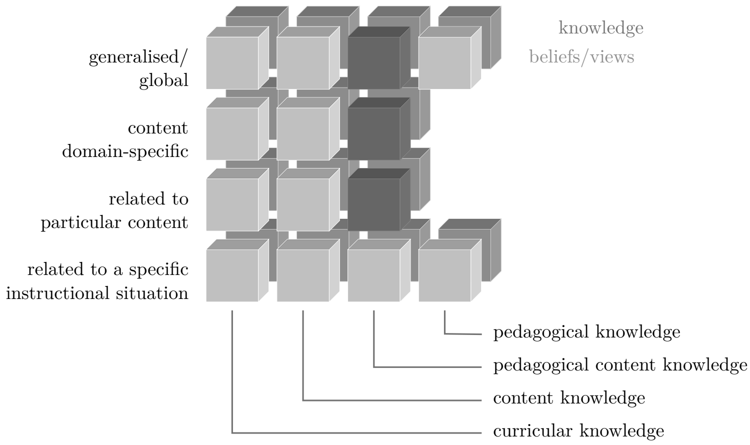


Figure 5.2: Overview model of components of professional knowledge (see Kuntze, 2012, p. 275)

It integrates three dimensions according to which different components of mathematics teachers' professional knowledge can be structured. Considering the difficulty of distinguishing knowledge from beliefs and views with respect to mathematics instruction (Lerman, 2001; Pajares, 1992; Pepin, 1999), in a pragmatic approach such beliefs are included as aspects of professional knowledge and consequently the spectrum between knowledge and beliefs constitutes one of those dimensions. A second dimension affords structuring aspects of professional knowledge according to the domains by Shulman (1986a) which form the basis of many recent models of teacher professional knowledge (e.g., Ball et al., 2008; Kunter et al., 2011). It is of course possible to refine these categories, for instance by using the domains of "Mathematical Knowledge for Teaching" suggested by Ball and colleagues (2008). Even if the cells in Figure 5.2 clearly have overlaps, an advantage of the model (Kuntze, 2012) used here compared to others lies in the possibility to structure components of professional knowledge also with respect to their globality (see Törner, 2002), which constitutes its third dimension. Knowledge and views about teaching mathematics can be very global – such as for instance beliefs about the discipline of mathematics – but they can also be specific to a certain content domain, a particular content or even to a particular instructional situation.

Regarding views on dealing with multiple representations in the mathematics classroom, the distinction of different levels of globality is very useful: Firstly, there are general views about the role that multiple representations play for pupils' learning in mathematics and in particular views on reasons for using multiple representations in the mathematics classroom. Perceptions of such reasons have probably a significant impact on how teachers design learning opportunities using multiple representations. For instance, seeing the main purpose of multiple representations in making mathematics instruction fun and diverse may serve the design of conceptually rich mathematical activities less than being aware of the fact that usually the interplay of different representations is needed for the development of an appropriate concept image. Secondly, content domain-specific views about how to deal with representations when teaching fractions merit further attention. As reasoned above, the development of a concept image of fractions which is sufficiently multi-faceted requires integrating and connecting multiple representations. Hence, focusing exclusively on one "standard" pictorial representation like "the pizza" does probably not foster deep conceptual understanding of fractions. However, such views about how to deal with multiple representations when teaching fractions might still be different from perceptions of what role multiple representations should play in particular tasks about fractions. So, thirdly, we focus on views about how multiple representations can foster pupils' learning in specific tasks, which may be seen as views related to a particular content. For instance, being aware of opportunities which can encourage pupils to actively create connections between representations of fractions and their operations may better support teachers in choosing and designing tasks with a high learning potential. In contrast, focusing on using pictorial representations of fractions for the sole purpose of encouraging pupils in engaging with the particular task, even if those representations are not useful for the solution, may be less helpful for designing conceptually rich learning opportunities. Fourthly, there are even more situated views about dealing with multiple representations, namely conceptions about how to use representations in specific instructional situations. All these views about dealing with multiple representations in the mathematics classroom on different levels of globality are considered to be part of a teacher's pedagogical content views. Those on the first three levels of globality are in the focus of this study and therefore, the corresponding components are highlighted in the model of professional knowledge shown in Figure 5.2. However, of course the most situated level of teachers' views should not be neglected. In a larger study about teacher professional knowledge, which forms the framework of the research presented here, we also address views about dealing with multiple representations of fractions in specific classroom situations (see Dreher & Kuntze, 2014).

Assessing views on different levels of globality affords exploring to what degree these views are interconnected consistently. This is particularly interesting, since there is some evidence suggesting that the development of professional knowledge for teaching mathematics is accompanied with a growth in consistency across levels of globality: Investigating how practicing teachers acquire professional knowledge, Doerr and Lerman (2009) have identified "the teachers' learning as a recurring flow between the procedural and the conceptual" (p. 439), where specific responses to problem situations were seen as pedagogical procedures and more general principles for actions as pedagogical concepts. "Procedural knowledge" is thus more situated and less global than "conceptual knowledge". Hence, the question as to what degree teachers' views on different levels of globality are consistent merits attention and might serve as an indicator for the development of professional knowledge. Inconsistency across levels of globality with respect to views on dealing with multiple representations could mean for instance that a teacher acknowledges in general that multiple representations should be connected for the pupils to develop an appropriate concept image, but nevertheless he or she thinks that teaching fractions works best when concentrating on a single pictorial representation. Consequently, research into teachers' professional knowledge – in this case into pre-service teachers' views on dealing with multiple representations in the mathematics classroom – should take into account different levels of globality and explore also to which degree these levels are interconnected or consistent. Yet, research into views on dealing with multiple representations in the mathematics classroom is scarce. There are to our knowledge only studies regarding some selected aspects of views on dealing with multiple representations, such as findings by Ball (1993a) concerning global beliefs of teachers about pictorial representations. These findings suggest that the interviewed (American) teachers attached great importance to the motivational potential of pictorial representations, whereas they rather neglected their role for conceptual learning. Another qualitative study has focused on teachers' expectations of pupils being able to perform different conversions of representations (Bossé et al., 2011). However, as reasoned above, there is a need for assessing views on dealing with multiple representations in a more multi-faceted manner taking into account global and more situated views as well as their interconnections in order to identify needs for professional development. It may be assumed that pre-service teachers even at the beginning of their teacher education already hold certain views about dealing with multiple representations which may in particular be shaped by their experiences as pupils. Therefore, analysing pre-service teachers' prerequisites regarding such views and their degree of consistency could support the design of appropriate learning opportunities for their teacher education.

The possible role of culture

Exploring teachers' views, one should bear in mind that some aspects may be culture-dependent (Pepin, 1999). The results of Pepin's qualitative research into epistemologies, beliefs and conceptions of mathematics teaching and learning in England, France, and Germany suggest that teachers' beliefs were influenced by their cultural environment. Her findings include for instance that for English teachers an individualistic and child-centred view was dominant, whereas the conception of mathematics revealed by the investigated German teachers was relatively formal. These global tendencies might also become evident in teachers' perceptions of the role that multiple representations play in the mathematics classroom. Since pre-service teachers' views at the beginning of their teacher education may in particular be influenced by their experiences as pupils, the investigation of their views should be seen in the light of characteristics of mathematics teaching in their countries. Kaiser (2002) pointed out the strong influence of different educational philosophies in England and Germany on the mathematics classroom. Based on her ethnographic study she described typical aspects of mathematics teaching in England and Germany in a contrasting way, which are briefly summarised in the following. According to Kaiser's study, the most important principle of the English education philosophy is the high priority of the individual. Hence, in the English mathematical classroom long phases of individual work are typical, where great emphasis is put on the pupils' own ways of problem solving with openness towards individual way of notation and formulation. In Germany however, class discussion in which ideas are developed collectively is usually a dominant teaching-and-learning style and thus a precise mathematical language which is comprehensible by all learners and common notation is seen as being more important. Following Kaiser (2002), distinct characteristics of mathematics teaching in England and Germany can furthermore be put down to contrasting understandings of the role of theory for teaching mathematics. The predominantly scientific understanding of theory in Germany typically leads to great significance of rules, formulae, and arithmetic algorithms, which often have to be learned by heart, whereas a rather pragmatic understanding of theory for teaching mathematics in England goes along with a focus on work with examples and minor relevance of rules and standard algorithms.

Bearing in mind these findings, comparing views of pre-service teachers from England and Germany on dealing with multiple representations may give some insight into which aspects of such views and corresponding needs might be rather culture-dependent versus culture-independent. This approach can also give feedback about the research instrument used with respect to its culture-sensitivity and validity.

The possible role of specific CK

It may be assumed that specific CK is interrelated with views on dealing with multiple representations, in particular when it comes to tasks-specific views, since evaluating the learning potential of a task requires a content-specific understanding of the given representations and their interplay. Especially the ability to see and reflect connections between different representations and – particularly in the case of fractions – to match symbolic-numerical representations with appropriate pictorial and content-related representations appears thus to be relevant. Reviewing the released items (Ball & Hill, 2008) of the survey instrument developed by Hill and colleagues (2004) in order to measure mathematical knowledge for teaching shows that there are items included which assess this special kind of CK. However it was not conceptualised separately in that study. Hill and colleagues described a way of categorising their items based on different sorts of teachers' tasks, where "choosing representations" is one of them (Hill et al., 2004), but in the light of the reasoning above, "choosing representations" does not capture the full spectrum of what is involved in dealing with multiple representations in the mathematics classroom. Although the COACTIV study (Kunter et al., 2011), which also focuses on teacher professional knowledge, has included "explaining and representing" as one of three components in its model for pedagogical content knowledge, as far as CK is concerned, representations appear not to play any explicit role in the research design. Consequently, there is a need for developing test instruments assessing specific CK regarding connections between multiple representations.

Research interest

According to the need for research pointed out in the previous section, the study presented here aims to provide evidence for the following research questions:

1. What views do English and German pre-service teachers have on the role of multiple representations for learning mathematics? In particular:
    (a) How much importance do they attach to different (global) reasons for using multiple representations in the mathematics classroom?
    (b) What (content domain-specific) views about how to deal with representations when teaching fractions do they have?
    (c) What (content-specific) views related to the learning potential of tasks focusing on conversions of representations, in comparison with tasks including rather unhelpful pictorial representations do the pre-service teachers have?
2. To what degree are their views on those different levels of globality interrelated? Is there evidence of inconsistencies, which may point to needs for their teacher education?
3. Do inter-cultural comparisons reveal any differences regarding such views and corresponding needs, i.e.: Are there any indications for which aspects might be rather culture-dependent than culture-independent?
4. What specific CK about dealing with multiple representations in the domain of fractions do English and German pre-service teachers have? Is this CK interrelated with their task-specific views on dealing with multiple representations?

Sample and methods

For answering these research questions, a corresponding questionnaire was designed in German and was then translated into English. This translation was examined carefully by two native speakers of English, one of whom is also fluent in German and has taught mathematics both in the UK and in Germany. The questionnaire is based on a previous version which was tested in a pilot study (see Kuntze & Dreher, 2014) and subsequently developed further. At the beginning of the questionnaire there were explanations of the notions *representation* and *pictorial representation* in a mathematical context given in order to reach a similar understanding of these key terms for the study.

The questionnaire was administered to 139 English (99 female, 22 male, 18 without data) and 219 German (183 female, 26 male, 10 without data) pre-service teachers before the beginning of a course at their university. The English participants had a mean age of 27.9 years (SD = 6.9), while the German participants were on the average 20.7 years old (SD = 2.5), but (with only a few exceptions in both samples) all the participants were at the beginning of their first year of teacher education at university. The age difference between the English and the German pre-service teachers in our study is due to the different systems of teacher education in the two countries. While in the UK pre-service teachers have already finished a university degree before starting a one-year teacher training, German pre-service teachers enter their university studies directly with a 4 to 5 year teacher education program. The pre-service teachers of both countries were preparing to teach at primary level.

Corresponding to the research questions for this study four parts of the questionnaire were included in the evaluations: The first three parts were assessing views on using multiple representations on different levels of globality and moreover there was a section about the participants' specific CK, namely their ability to match symbolic-numerical representations of fractions with appropriate pictorial and content-related representations. In the following, these four questionnaire sections are described in more detail.

In order to explore which reasons for using multiple representations in the mathematics classroom are most important to English and German pre-service teachers, the participants were asked to rate the significance of possible reasons on a five-point Likert scale (from *not important* to *extremely important*). The selection of four different types of reasons which are shown in Table 5.2 is not only drawn from theoretical considerations and literature review, but is also based on the results of a pilot study in which the participants could give reasons for using multiple representations in the mathematics classroom in an open format (see Dreher, 2012a). The appropriateness of the corresponding model encompassing four constructs was examined empirically be means of a confirmatory factor analysis (CFA) which is reported in the results section.

Construct (identifier)	Sample item	# items
Necessity for mathematical understanding	Enhancing the ability to change from one representation to another is essential for the development of mathematical understanding.	4
Motivation and interest	They make it easier to keep pupils' interest.	3
Supporting remembering	Pupils can use pictorial representations as mnemonics.	3
Learning types and input channels	Different learning types and input channels can be addressed.	3

Table 5.2: Scales regarding reasons for using multiple representations in the mathematics classroom

Based on the above reasoning about domain-specific views on the role of multiple representations for teaching fractions a corresponding questionnaire section was designed which focuses on the five constructs presented in Table 5.3. With respect to each item the pre-service teachers could express their approval or disagreement on a four-point Likert scale. The structure of our proposed model of five constructs was again examined with respect to the empirical data by means of a CFA.

Construct (identifier)	Sample item	# items
Multiple representations (MR) for understanding	To understand fractions properly, it is necessary to use many different representations in class.	3
Multiple representations (MR) for individual preferences	In order to give pupils the opportunity to choose their preferred type of representation, which they most easily understand, they should be provided with many different representations.	3
One standard representation	It is best to use only one kind of pictorial representation for fractions in lessons, so that you can always come back to this as a 'standard' representation.	3
Fear of confusion by multiple representations (MR)	Several different pictorial representations for fractions could confuse pupils, especially the weaker ones.	3
Multiple representations (MR) impede learning rules	If pupils pay too much attention to pictorial representations, their ability to confidently do calculations with fractions is impeded.	3

Table 5.3: Scales regarding views on dealing with multiple representations for teaching fractions

To explore if English and German pre-service teachers are able to recognise the learning potential of tasks focusing on conversions of representations, in comparison with tasks including rather unhelpful pictorial representations, the participants were asked to evaluate the learning potential of six fraction tasks by means of three multiple-choice items. A sample item is: "The way in which representations are used in this problem aids pupils' understanding." The pre-service teachers could express their approval or disagreement concerning these items regarding each task on a four-point Likert scale. They were told that the tasks were designed for an exercise about fractions in school year six. Three of these tasks are about carrying out a conversion of representations, whereas solving the other three tasks means just calculating an addition or a multiplication of fractions on a numerical-symbolical representational level. The pictorial representations which are given in the tasks of this second type are rather not helpful for the solution, since they cannot illustrate the operation needed to carry out the calculation. Samples for both kinds of tasks are shown in Figure 5.3.

Make up a situation or a word problem which is suitable for the calculation $3 \div \frac{1}{4}$ and then use it to solve the calculation.

Do you know what $\frac{1}{2} \times \frac{1}{4}$ is? You can use the pictures below to help:

Figure 5.3: Samples for tasks of type 1 (left) and of type 2 (right)

Obviously, these two types of tasks are not representative of all tasks about fractions and moreover there are of course other kinds of fraction tasks that have a high learning potential. However, the idea behind this rather plain, bipolar design is that contrasting these two types of tasks against each other affords insight into whether and where the pre-service teachers see the learning potential of multiple representations for fraction tasks. The assumption that the theoretical classification of the tasks underlying their creation manifests itself also in the pre-service teachers' evaluations of their learning potential suggests that two second order factors can be empirically separated which represent the evaluations of the types of tasks. The analysis of how well this theoretical model fits the empirical data was carried out by a CFA and is reported in the results section.

The last questionnaire section which is included in this study focuses on special CK about dealing with multiple representations in the domain of fractions. More specifically, this CK test was designed to assess the participants' ability to match symbolic-numerical representations of fractions and their operations with appropriate pictorial and content-related representations. As in the sample item shown in Figure 5.4, given (incorrect) conversions between such representations had to be checked and corrected or a conversion had to be carried out.

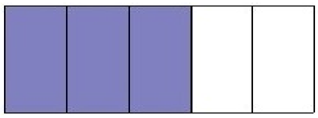
Please change the diagram, if necessary, so that $\frac{3}{5}$ of $\frac{1}{4}$ is shaded. Otherwise just tick the box on the right-hand side.		

Figure 5.4: Sample item of the CK test

Results

We start with the results concerning the pre-service teachers' rating of the importance of reasons for using multiple representations in the mathematics classroom. In order to examine the theory-based structure of the corresponding questionnaire instrument (cf. Table 5.2) empirically, a confirmatory factor analysis (CFA) was conducted on the items in this Section. The appropriateness of the theoretical model was assessed by several measures of global model fit. Firstly, measures of incremental fit were employed, namely the *TLI* (Tucker-Lewis index) and the *CFI* (comparative fit index). In both cases an acceptable fit is indicated by values ≥ 0.90 (Kline, 2005). For the proposed model the *TLI* is 0.93 and the *CFI* is 0.95 and hence both criteria are met. Moreover, we examined the RMSEA (root mean square error of approximation), which can be interpreted as the amount of information within the empirical covariance matrix not explained by the proposed model. The model may be classified as acceptable if at most 8 % of the information are not accounted for by the model, i.e., *RMSEA* ≤ 0.08 (Kline, 2005). The current model meets this criterion as *RMSEA* is 0.064. Thus, it was concluded that the model fits the data reasonably well. Concerning the local model fit, the analysis showed that all factor loadings are highly significant ($p < .001$) and that the factor reliabilities range from .68 to .79. In order to determine whether the four constructs of the proposed model are empirically separable, the discriminant validity of each construct with respect to the others was assessed by means of chi-square difference tests (see e.g., Jöreskog, 1971). More precisely, it was examined for each pair of constructs in the model, whether restricting their correlation to 1 leads to a significantly poorer data fit. Since in each case this parameter restriction caused a significantly poorer fit ($p < .001$), it was concluded that the four constructs of the proposed model show sufficient discriminant validity. Therefore, for each of the four constructs a corresponding scale could be formed.

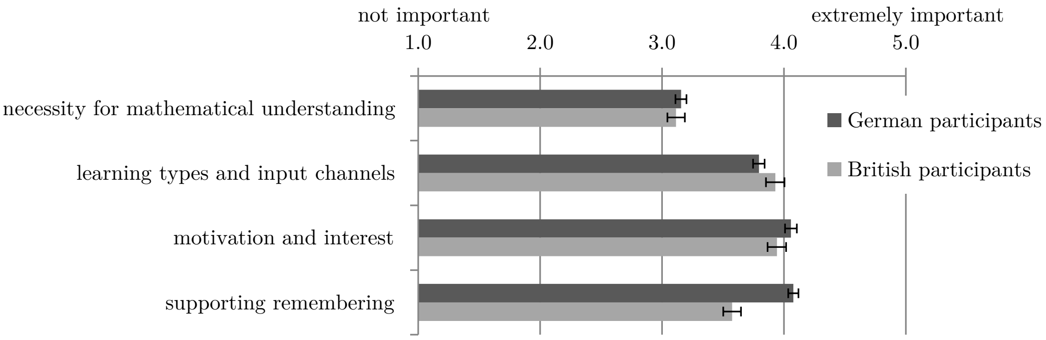


Figure 5.5: Views on the importance of reasons for using multiple representations

Figure 5.5 shows the means and their standard errors for these scales for both subsamples. First, it is noticeable that both subsamples rated the discipline-specific reasons as less important than the other more general reasons. Furthermore there are no significant differences between the ratings of the English and the German pre-service teachers, except for the last scale: The German pre-service teachers attributed a higher significance to the contribution of multiple representations to remembering mathematical facts than did their English counterparts ($t(207) = 6.0$, $p < .001$, $d = 0.73$).

Regarding the questionnaire section about views on the role of multiple representations for teaching fractions we also started by examining whether our theoretical model fits the empirical data. Conducting a CFA, the model exhibited a reasonably good data fit (*RMSEA* = 0.054, *TLI* = 0.92, *CFI* = 0.95). All the factor loadings are highly significant and the factor reliabilities range from .62 to .76. Furthermore, conducting chi-square difference tests as described above suggested that the five constructs in the proposed model have sufficient discriminant validity. Consequently, five scales corresponding to the five constructs could be formed.

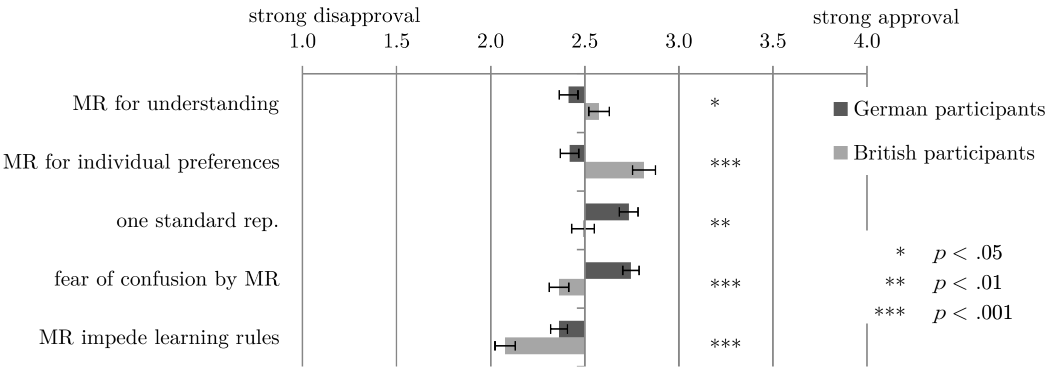


Figure 5.6: Views on the role of multiple representations for teaching fractions

Comparing the means of the two subsamples shown in Figure 5.6, one discovers an interesting pattern: the English pre-service teachers compared to the German pre-service teachers were more in favour of using multiple representations for teaching fractions and less afraid of possible negative effects. Cohen's d shows that the difference concerning the scale "multiple representations for understanding" is rather negligible ($d = 0.25$), whereas the other significant differences represent weak or medium effects ($0.35 < d < 0.58$). The scales which detected the biggest differences between the English and the German participants are "MR for individual preferences" and "fear of confusion by MR".

Going down another level of globality, we focus now on the pre-service teachers' evaluations of the learning potential of the six tasks about fractions given in the questionnaire. Since this evaluation was carried out by means of three items regarding each task and since these tasks were in turn designed to represent two different types of tasks ("conversions of representations" vs. "unhelpful pictorial representation") the proposed model for this questionnaire section encompasses $6 \times 3 = 18$ indicators of six first order factors (the evaluations of the tasks) and two second order factors (the evaluations of the types of tasks). A CFA yielded a reasonably good data fit for this model ($RMSEA = 0.054$, $TLI = 0.92$, $CFI = 0.93$). As regards the local model fit, the factor loadings are all highly significant and both second order factors are reliable with $\alpha = 0.75$ and $\alpha = 0.73$. For assessing the discriminant validity of the two constructs "evaluation of the learning potential first type of tasks" and "evaluation of the learning potential of the second type of tasks", a chi-square difference test was carried out. The fact that this test was highly significant ($p < .001$) indicates that the model which distinguishes between the pre-service teachers' evaluations of the two types of tasks predicts the empirical data better than the corresponding one-dimensional model. Therefore, two scales corresponding to the evaluations of the two types of tasks were formed. The means and their standard errors of these scales in Figure 5.7 show that both the English and the German pre-service teachers tended to rate the learning potential of the tasks of type 2 on average slightly higher than the learning potential of those of type 1. However, the difference is merely significant with respect to the English subsample ($t(138) = 2.7$, $p = .008$, $d = 0.24$).

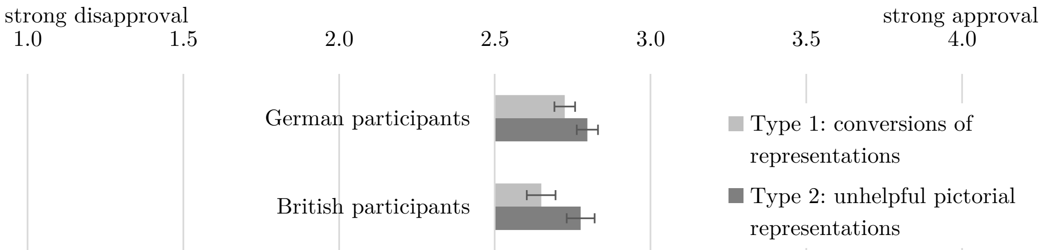


Figure 5.7: Evaluations of the learning potential regarding the two types of tasks

So far we reported results concerning pre-service teachers' views on dealing with multiple representations on three levels of globality. Addressing our second research question, we focus now on relationships between these levels in order to explore to what degree the corresponding views are consistent. The idea behind the following analysis is basically to explore whether the pre-service teachers' general conceptions about the role that multiple representations play for pupils' learning in mathematics "translate" into corresponding content- and task-specific views. Seen against the theoretical background of our study, a key idea is that learning with multiple representations is essential for the development of appropriate mathematical concept images and therefore for conceptual understanding of mathematics. This view is reflected in the scale "necessity for mathematical understanding" which is listed in Table 5.2. Under the assumption that this global conception transfers into corresponding content domain-specific views, one would particularly expect that it is interrelated with the view that fractions should be taught using multiple representations for the sake of the pupils' understanding. Furthermore, one would expect that a higher approval of such discipline-specific reasons for using multiple representations is related to a higher rating of the learning potential of the tasks encouraging pupils to create connections between different representations. Consequently, we examine whether the corresponding scales correlate. Pearson's correlation coefficients are presented in Table 5.4 for both subsamples separately.

First, it can be noted that for both subsamples the assigned significance to the reason "necessity for mathematical understanding" on a global level is indeed positively correlated with the content-specific view that for teaching fractions multiple representations support the pupils' understanding. In both cases the corresponding correlation coefficients represent moderate effect sizes. However, looking deeper into the data in order to find reasons why these correlations are not higher still, we found that in both subsamples there are participants

English participants	MR for understanding (fractions)	Type 1	Type 2
Necessity for understanding	.41**	.12ns	.38**
German participants	MR for understanding (fractions)	Type 1	Type 3
Necessity for understanding	.39**	.21**	.14*
ns = not significant ($p \geq .05$), * $p < .05$, ** $p < .01$			

Table 5.4: Pearson's correlation coefficients for the scale "necessity for mathematical understanding"

whose views on these two levels of globality are not consistent at all. For instance, one German pre-service teacher has rated the significance of the general reason for using multiple representations "necessity for mathematical understanding" very high (4.75), whereas he showed very little approval of the corresponding content-specific view regarding fractions (1.33).

Focusing on interrelations with the perceived learning potentials of the two types of fraction tasks, our subsamples appear to be more distinct (cf. Table 5.4). Merely for the German subsample the assigned importance to the discipline-specific reasons correlates slightly positively with the perceived learning potential of the type 1 tasks ($r = .21^{**}$). With respect to the English participants it correlates instead positively with the perceived learning potential of the calculation tasks of type 2 with unhelpful pictorial representations ($r = .37^{**}$). For the German subsample there was also such a correlation found, which is however barely significant and represents a weak effect ($r = .14^{*}$). This raises the question whether the pre-service teachers have recognised that those pictorial representations in the type 2 tasks cannot illustrate the operation needed to carry out the calculations. Thus, we next address the fourth research question and focus on the results regarding the specific CK test included in the questionnaire.

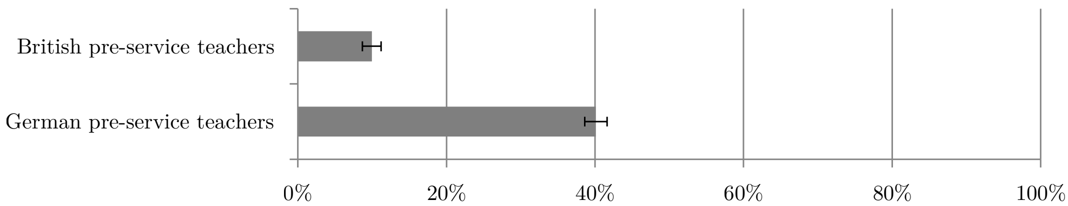


Figure 5.8: Specific CK scores (means and their standard errors)

Figure 5.8 presents the mean scores (and their standard errors) of the two subsamples for this test. It appears obvious that the German participants achieved significantly higher scores than their English counterparts ($t(355) = 15.4$, $p < .001$, $d = 1.5$). However, the English as well as the German pre-service teachers in our sample solved on average less than half of the items correctly, which indicates a common need for development of professional knowledge, in this case CK. This might suggest that the participants in this study did not have enough specific CK for evaluating the use of representations in tasks of type 2 appropriately. Yet, no significant correlation between the pre-service teachers CK scores and their evaluation of type 2 tasks was found.

Discussion and conclusions

The findings of this study about aspects of English and German pre-service teachers' views on the role of multiple representations for learning mathematics affords identifying prerequisites and specific needs for initial teacher education as well as insight into culture-dependent facets of such views. Before discussing these results in more detail, we would like to recall the limitations of this study which suggest interpreting the evidence with care. Despite the size of the subsamples, the study is not representative for German or English pre-service teachers. Moreover, although a spectrum of different facets of views on dealing with multiple representations was included in the design, the constructs can only give an indicator-like insight and are mostly restricted to the domain of fractions. Bearing this in mind, the findings allow however to answer the research questions and indicate several aspects of theoretical and practical relevance. We start by discussing the findings regarding the first two research questions with a focus on common aspects of the views and prerequisites of the pre-service teachers from England and Germany. Both subsamples saw on average the special role of multiple representations for understanding mathematics as less important than the other reasons for using multiple representations, which are not discipline-specific. Moreover, the English as well as the German participants were mostly not able to recognise the learning potential of tasks focusing on conversions of representations, in comparison with tasks including rather unhelpful pictorial representations, to which the pre-service teachers tended to assign a higher learning potential. This demonstrates that the pure global conviction of "using multiple representations is good" is not enough for designing rich learning opportunities, but in addition answers to questions such as "for what purpose and in which way should multiple representations be used" are central.

Regarding the second research question about connections between the pedagogical content views, the evidence suggests some expected interrelations of the pre-service teachers' views on different levels of globality. However, those interrelations are not strong, so it may not be assumed that global views simply "translate" into content-specific views, but that the views on the different levels of globality represent constructs of their own right. Considering cases of participants whose views on different levels of globality appear to be even contradictory, reinforces the impression that for many pre-service teachers, global views on dealing with multiple representations in the mathematics classroom were not (yet) very consistent with their corresponding domain- and task-specific views.

It may not be surprising that beginning pre-service teachers show little awareness for the special role that representations play for mathematical understanding and that their views are not (yet) very consistent. Nevertheless these findings provide insight into the specific prerequisites and needs of these pre-service teachers and afford to customise their professional development to the end of fostering their professional knowledge with respect to dealing with multiple representations in the mathematics classroom. Regarding such common needs for initial teacher education, we can draw the following conclusions: Firstly, awareness of the crucial role of multiple representations and their connections for conceptual understanding of mathematics should be seen as a key element in the development of pedagogical content knowledge. Secondly, the work on specific content, tasks and also instructional situations should be in the centre of professional learning under the perspective of overarching ideas such as using multiple representations (see Kuntze et al., 2011), in order to support pre-service teachers develop specific pedagogical content knowledge which is consistent with respect to different levels of globality.

Besides these common prerequisites and needs for teacher education, some of the findings yielded differences between the English and the German subsample which indicate culture-dependent aspects of views on dealing with multiple representations. In line with the third research question, looking at such differences also affords designing opportunities for professional learning which may be more valid within the framework of the specific cultural settings. With respect to reasons for using multiple representations in mathematics classrooms in general, the only significant difference we identified was the greater emphasis of the German pre-service teachers on remembering facts. Regarding content-domain specific views related to the use of multiple representations, however, more differences became apparent. For teaching fractions, the English pre-service teachers attached significantly greater importance to multiple representations than their German counterparts – at least when reasons not specific to mathematics were in the focus. The German pre-service teachers rather feared confusing their pupils by multiple representations and favoured the use of one 'standard' representation more than did their English counterparts. The German pre-service teachers may hence have put a focus mainly on learning rules, whereas for the English participants taking into account individual preferences was predominant. Interestingly, these differences in the views expressed by the English and German pre-service teachers in this study reflect very well the differences in the teaching and learning styles in England and Germany as they were described by Kaiser (2002). This can in particular be seen as a further validation of the questionnaire instrument used in this study. Moreover, our results are also consistent with the findings by Pepin (1999) regarding more general views of teachers in England and Germany and they may in particular add PCK-specific aspects to these findings.

However, in order to draw conclusions from this culture-related evidence, the content-specific views regarding the fraction tasks should be included in order to provide a more complete picture. For instance, correlations of the investigated task-specific views with the perceived significance of discipline-specific reasons for using multiple representations indicate inconsistencies across levels of globality, specifically for the English participants. Greater appreciation of the role of using multiple representations for building up conceptual understanding was on average associated with a more positive evaluation of the learning potential of calculation tasks with rather unhelpful pictorial representations (cf. Table 5.4). Moreover, the on average very low scores in the specific CK test of the English participants may indicate that most of them did not realise that those pictorial representations were not helpful, but merely noticed that there were different representations provided. Hence, these pre-service teachers may need a strengthened CK background; teacher education should combine specific help in that area with learning opportunities connected to content-specific PCK.

We could not observe any direct correlation between specific CK and the pre-service teachers' task-specific views. Nevertheless, corresponding CK might be of increased significance for the teachers' ability to implement the global goal of fostering the learners' understanding with multiple representations, as representations and their interrelations must be analysed accurately on the content level. Hence, this calls for deepening studies which explore the role that domain-specific CK plays regarding domain-specific views about how to deal with multiple representations. In particular, further research should also include in-service teachers, since this may give further evidence for the hypothesis that the development of professional knowledge for teaching mathematics is accompanied with a growth in consistency across levels of globality. Moreover, such extended research should focus on the most situated level of globality as well and examine teachers' professional knowledge and views on dealing with multiple representations regarding specific classroom situations (cf. Dreher & Kuntze, 2014). In addition, deepening studies should use qualitative methods in order to illustrate different profiles of professional knowledge about dealing with multiple representations and analyse them in greater depth.

5.2 Article 2

Teachers' professional knowledge and noticing – The case of multiple representations in the mathematics classroom

Anika Dreher, Sebastian Kuntze

Abstract
Teachers notice through the lens of their professional knowledge and views. This study hence focuses not solely on teachers' noticing, but also on their knowledge and views, which allows insight into how noticing is informed and shaped by professional knowledge. As a discipline-specific perspective for noticing we chose dealing with multiple representations, since they play a double role for learning mathematics: On the one hand they are essential for mathematical understanding, but on the other hand they can also be an obstruction for learning. This comparative study takes into account pre-service as well as in-service teachers in order to explore the role of teaching experience for such professional knowledge, views and noticing. The participants answered a questionnaire addressing different components of specific knowledge and views. For eliciting the teachers' theme-specific noticing, vignette-based questions were used. The data analysis was done mainly by quantitative methods, but was complemented by a qualitative in-depth analysis focusing on how the teachers' theme-specific noticing was informed by different components of their professional knowledge. The results suggest that pre-service as well as in-service teachers do not fully understand the key role of multiple representations for learning mathematics in the sense of their discipline-specific significance. The participating in-service teachers distinguished themselves however from the pre-service teachers especially regarding their theme-specific noticing. Moreover, the evidence indicates that teachers' noticing of critical instances of dealing with multiple representations draws on situated as well as on global knowledge and views.

Keywords: Teacher professional knowledge, Teacher noticing, Multiple representations, In-service teachers, Pre-service teachers

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Introduction

Among researchers in mathematics education multiple representations and their role for learning mathematics have received a lot of attention, especially during the last 30 years (e.g., Acevedo Nistal, van Dooren, Clareboot, Elen, & Verschaffel, 2009; Ainsworth, 2006, Duval, 2006; Goldin & Shteingold, 2001; Janvier, 1987; Kaput, 1989). As a result, there is broad acceptance in the scientific community that dealing with multiple representations is an essential as well as delicate issue for teaching and learning mathematics. While there is a substantial basis of empirical research focusing on students' learning with multiple representations (e.g., Acevedo Nistal et al. 2009; Ainsworth, 2006; Renkl, Berthold, Große, & Schwonke, 2013), there are merely a few studies addressing the role of the teachers in this context. In this study we thus raise the question of how much teachers know about and acknowledge this key role of multiple representations for the mathematics classroom. To this end we explore particularly their noticing of changes of representations in instances of student-teacher interaction, which can be seen as a theme-specific noticing. Focusing however not only on teachers' noticing, but also on their professional knowledge and views allows us to explore interrelations of noticing on the one hand and knowledge and views on the other hand and hence responds to a research gap pointed out by Schoenfeld (2011). Using a multi-layer model of professional knowledge we address in particular the question as to what components of knowledge and views inform teachers' theme-specific noticing. For the content-specific parts of this study we chose the domain of fractions, given the high relevance of using multiple representations specifically in this content domain (e.g., Ball, 1993a; Prediger, 2011). The design of the study includes pre-service as well as in-service teachers in order to get insight into potential differences between expert and novice teachers regarding their views and knowledge about dealing with multiple representations and their specific noticing.

The following first section gives an overview of the theoretical background, which leads to the research interest for this study as presented in the second section. We will then describe the design and methods of the study in the third section, report results in the fourth section and conclude with a discussion in the fifth section.

Theoretical background

In this section, we first focus on the key role that representations and changing between them play for students' learning in the mathematics classroom. In the second part consequences for teaching mathematics are deduced and corresponding pedagogical content knowledge (PCK) is identified in the context of a model for teacher professional knowledge. We emphasize the need to distinguish between different levels of situatedness regarding such teacher knowledge and views and to take into account their roles for noticing. The relationship between noticing on the one hand and knowledge and views on the other hand is discussed in the last part of this section and corresponding needs for research are deduced.

Multiple representations in the mathematics classroom

Doing mathematics relies on using representations, since mathematical objects are not accessible without them (Duval, 2006; Janvier, 1987; Mason, 1987). Figure 5.9 shows an example which illustrates this phenomenon: Some possible representations for the fraction $\frac{2}{3}$ are given, but none of them is the fraction itself. They can merely stand for the mathematical object (cf. Duval, 2006; Goldin & Shteingold, 2001) and make visible different aspects and characteristics of it. Hence, in order to see different facets of the corresponding mathematical object and to develop an appropriate concept image, usually several of such representations have to be integrated (Ainsworth, Bibby, & Wood, 1998; Duval, 2006; Even, 1990; Goldin & Shteingold, 2001; Janvier, 1987; Tall, 1988).

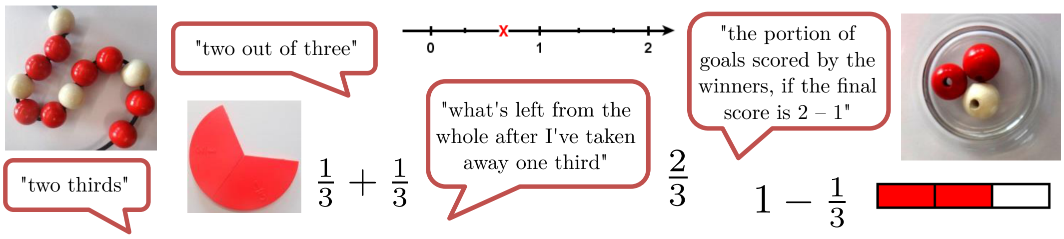


Figure 5.9: Some representations for the fraction $\frac{2}{3}$

Consequently, representing mathematical objects in multiple ways plays an important role for mathematical understanding (Ainsworth et al., 2002; Duval, 2006; Elia, Panaoura, Eracleous, & Gagatsis, 2007; Even, 1998). However, a representation does not stand for a mathematical object in any obvious, self-explanatory way. Instead, this connection is subject to interpretation and negotiation processes (Cobb, 2002; Gravemeijer, Lehrer, van Oers, & Verschaffel, 2002; Meira, 1998). Thus, every time students are introduced to a new representation, they must learn how it is used and interpreted in the mathematics community and in their mathematics classroom. Moreover, it is not enough to consider this representation in an isolated way for not confusing it with the corresponding mathematical object, but connections have to be made with other representations of this object in order to go beyond the specific representation and to be able to change between different representations (Duval, 2006; Even, 1998; Kaput, 1989).

These cognitive processes are usually highly demanding for learners and can pose an obstacle to comprehension (Ainsworth, 2006; Duval, 2006; Sfard, 2000). Multiple representations hence play an ambiguous role for learning mathematics: On the one hand they are essential for the construction processes of mathematical understanding and the ability to deal with them flexibly is key to successful mathematical thinking and problem solving (e.g., Acevedo Nistal et al. 2009; Lesh, Post, & Behr, 1987; Stern, 2002; Zbiek, Heid, & Blume, 2007). On the other hand multiple representations can function as an obstacle for learning mathematics, since interpreting them, recognizing their connections and changing between them are challenging tasks – and yet necessary for benefitting from them (e.g., Ainsworth, 2006; English & Halford, 1995; Janvier, 1987). There is substantial empirical evidence for this phenomenon: Using multiple representations can foster students' learning, but only if the students are encouraged to actively create connections between these representations (Bodemer & Faust, 2006; Rau, Aleven, & Rummel, 2009; Rau, Aleven, Rummel, & Rohrbach, 2012; Renkl et al. 2013). Rau et al. (2009) for instance conducted a study with so-called intelligent tutoring systems and found that students learned more with multiple pictorial (i.e., graphical) representations of fractions than with a single pictorial representation – but only when prompted to self-explain how the pictorial representations relate to the symbolic fraction representations.

This key role of multiple representations for learning mathematics, which was and still is emphasized by many researchers in the field of mathematics education, is also reflected in the present national standards of many countries (cf. e.g., KMK, 2003; NCTM, 2000). The German national standards for instance highlight "using mathematical representations" as one out of six general aspects of mathematical competence, which includes "applying, interpreting, and distinguishing different representations for mathematical objects and situations", "recognizing connections between representations" and "choosing different representations depending on the situation and purpose and changing between them" (cf. KMK, 2003, p. 8, translation by the authors). The characterization of "using mathematical representations" as a general aspect of mathematical competence refers to the fact that it applies to all kinds of mathematical contents. It has, however, content-specific facets as well, since different content domains typically emphasize different kinds of representations, which have to be used and integrated by the learners (e.g., Acevedo Nistal et al. 2009; Graham, Pfannkuch, & Thomas, 2009; Kuhnke, 2013). Essential representations in the content domain of fractions for instance highlight different core aspects of the concept of fractions such as part-whole, ratio, operator, quotient, and so on (e.g., Charalambous & Pitta-Pantazi, 2007; Malle, 2004). For this content domain it is particularly well-known that the development of an appropriate multi-faceted concept image of fractions requires integrating and connecting these (multiple) representations (e.g., Ball, 1993a; Brenner, Herman, Ho, & Zimmer, 1999; Siegler et al., 2010). Furthermore, fostering students' abilities to match – in particular – symbolic-numerical representations with appropriate pictorial (diagrams, sketches, illustrations) and content-related representations such as real world situations plays a key role for sustainable learning of operations with fractions (e.g., Malle, 2004; Prediger, 2011; cf. also Stern, 2002). For these reasons we chose to focus on the content domain of fractions for the domain-specific parts of this study.

Bearing in mind the findings about learning mathematics with multiple representations, consequences for teaching mathematics can be deduced. Firstly, learning environments should give students the opportunity to get to know different representations of a mathematical object in order for them to develop an appropriate multi-faceted concept image. Secondly, changing representations should not be done carelessly, but the students should be fostered explicitly to make connections and to reflect on conversions between representations, so that multiple representations do not become an obstacle to learning (Dreher, 2012b; Duval, 2006; Renkl et al., 2013). In order to deal with these requirements in the mathematics classroom, specific teacher knowledge is needed. Therefore, we address the question as to what specific knowledge and views teachers may have about how to use multiple representations for teaching mathematics.

Pedagogical content knowledge about dealing with multiple representations

For exploring such professional knowledge, this study uses the multi-layer model (Kuntze, 2012) which is illustrated in Figure 5.10.

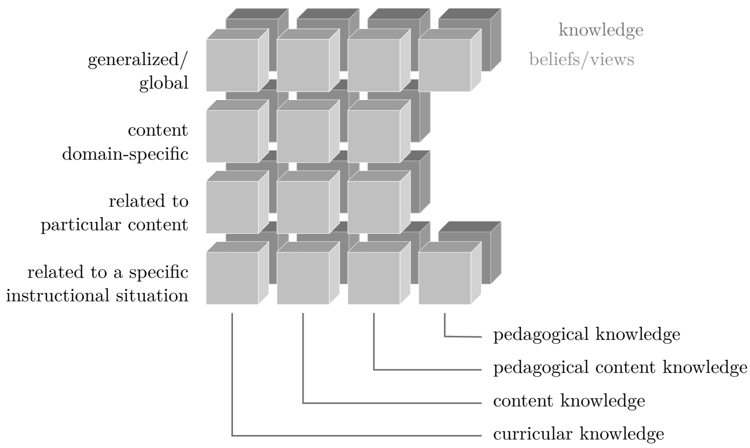


Figure 5.10: Overview model of components of professional knowledge (see Kuntze, 2012, p. 275)

It has three dimensions, with each of these dimensions helping to structure mathematics teachers' professional knowledge in a different way. Responding to the difficulty to differentiate between knowledge and beliefs regarding mathematics, mathematics instruction and pedagogy (Lerman, 2001; Pajares, 1992; Pepin, 1999), this model pragmatically integrates beliefs and convictions as aspects of professional knowledge. The spectrum between knowledge and beliefs thus forms one of these dimensions. When we use the notions *knowledge* and *beliefs/ views* in the following, one should hence keep in mind that we do not see them as being strictly separable. A second dimension reflects the possibility to structure teacher professional knowledge according to the domains identified by Shulman (1986a), which lie at the heart of many recent models of teacher professional knowledge (cf. Depaepe, Verschaffel, & Kelchtermans, 2013). These models also offer ways to refine these categories, for instance by using the domains of Mathematical Knowledge for Teaching suggested by Ball, Thames, and Phelps (2008). It is important to note again that even though in Figure 5.10 the cells look clearly separated – they overlap in fact, since the corresponding components of professional knowledge cannot be strictly separated (Kuntze, 2012). Nevertheless, these possibilities to distinguish between different components of teacher knowledge have proven useful – in particular also structuring components of professional knowledge with respect to their globality versus situatedness (cf. Törner, 2002), which constitutes the third dimension of the model (Kuntze, 2012). In the following, we briefly illustrate how this distinction of different levels of globality can be used in the case of pedagogical content knowledge and views about dealing with multiple representations in the mathematics classroom:

Firstly, there are general views and knowledge about the role that multiple representations play for students' learning mathematics and in particular views on reasons for using multiple representations in the mathematics classroom in general. Some teachers may be aware of the fact that successful mathematical thinking usually depends on the interplay of different representations, whereas others may for instance see the main purpose of multiple representations in making mathematics instruction fun and diverse (cf. Kuntze & Dreher, 2014).

Secondly, content domain-specific knowledge and views about how to deal with representations, for example, when teaching fractions, merit further attention. As reasoned above, the development of a concept image of fractions which is sufficiently multi-faceted requires integrating and connecting different representations. Focusing exclusively on one "standard" pictorial representation such as "the pizza" is therefore not likely to foster a high conceptual understanding of fractions.

Such content domain-related knowledge about how to use multiple representations best for teaching fractions might still be different from knowledge and views related to particular content and tasks. Corresponding questions are, for instance, how to use pictorial representations to explain the addition of fractions, or how to use multiple representations in specific fraction problems to create rich learning opportunities. Hence, a third level of globality regarding teacher knowledge about dealing with multiple representations which is connected to particular contents can be distinguished. These content-related professional knowledge components lie at the center of interest in another study that belongs to the same framework as the research presented here (cf. Dreher, Kuntze, & Lerman, submitted).

Besides the global, the content domain-specific and the content-related levels of globality, knowledge about how the use of multiple representations can foster or hinder students' understanding in particular instructional situations can play an important role. A common problem in the mathematics classroom is, for instance, that teachers often change between representations without really noticing it, since they already see the connections and hence the conversion seems to be obvious and can be done almost automatically (Duval, 2006; Gerster & Schulz, 2000; Kuhnke, 2013). For learners however, as reasoned above, such conversions of representations are usually highly demanding and they need to be supported to link and integrate these representations before they may eventually get to a point at which the conversion is obvious for them as well. In order to avoid such difficulties in the classroom, knowledge about the problematic nature of particular conversions of representations in certain situations may be needed. Such knowledge is closely tied to specific instructional situations; it may be connected to a teacher's individual classroom experience, especially in light of the fact that professional knowledge is often organized episodically (Leinhardt & Greeno, 1986).

For acting and reacting under the complex conditions of classrooms, knowledge such as that presented above is in turn still not sufficient. Teachers also have to focus their attention on corresponding significant aspects of what happens in the classroom and make connections to relevant knowledge (cf. Sherin, Jacobs, & Philipp, 2011) – for instance in order to recognize a “critical” change of representations as such.

Teachers' noticing and knowledge

In view of the framework developed by van Es and Sherin (2002), paying attention to changes of representations in specific instructional situations and evaluating whether they are sensible for students' learning can be seen as a form of theme-specific noticing. As the significance of teachers' noticing gets more and more attention in the field of mathematics education, there is a growing body of research concerning this matter, for which however somewhat different conceptualizations of noticing are used (Sherin et al., 2011). For this study we refer to the notion in the way it was specified by van Es and Sherin (2002). They emphasize that teachers' noticing is not only about attending to what is significant in a classroom situation, but that it also includes making sense of and reasoning about what is observed by drawing on corresponding knowledge.

Inspired by this theoretical framework and in view of the crucial role of multiple representations in the mathematics classroom, for our study we select a particular focus for teachers' noticing – namely the potentially obstructing demands of conversions of representation for students' understanding – and refer to it in the following as *theme-specific noticing*. Taking into account however not merely teachers' noticing, but also their knowledge and views, we are particularly interested in how noticing is interrelated with different components of professional knowledge. Thereby we respond to the need for studying the connections between teachers' noticing and their knowledge and views, which was pointed out by Schoenfeld (2011). He appealed to not treating teachers' noticing in an isolated way, but taking into account also their so-called resources and orientations among which are the components of professional knowledge outlined above:

> Noticing is essential, but it does not suffice by itself. It takes place within the context of teachers' knowledge and orientations; and the decisions that teachers make regarding whether and how to follow up on what they notice are shaped by the teachers' knowledge (more broadly resources) and orientations. (p. 233)

So teacher's noticing on the one hand and their knowledge and views on the other hand are somehow connected, but how is this relationship constituted? This is of course a far-reaching question, which is related to the more general question of how teachers' knowledge is interrelated with their actions and reactions in the classroom. One aspect however appears to be clear: The fact that noticing involves drawing on corresponding knowledge and views means that when a teacher notices successfully, we can draw conclusions regarding his or her professional knowledge and views. Also, there is broad agreement in the scientific community that teachers' noticing depends on the lens of their knowledge and views (e.g., Heid, Blume, Zbiek, & Edwards, 1999; Schifter, 2011; van Es, 2011). But what kind of professional knowledge is it that informs noticing? Is it global PCK in the sense of broader principles or rather content domain-related or even situation-specific knowledge and views or is it several of these knowledge components?

There is also good reason to assume that specific content knowledge (CK) is often a prerequisite for teacher noticing. In order to notice, for instance, a situation in the classroom in which it is appropriate to change representations, content knowledge is needed about different representations for the mathematical object at hand, about their connections and about the aspects they highlight. For gaining insight into the interplay of noticing and professional knowledge, it hence appears to be suitable to take into account not only teachers' PCK, but also their specific CK.

The ability to notice is often used to characterize expert teachers (e.g., Ainley & Luntley, 2007; Berliner, 1994; Jacobs, Lamb, & Philipp, 2010; Mason, 2002).With respect to our study with pre-service and in-service teachers this suggests that particularly when it comes to noticing, differences between these two groups should become visible. Noticing, however, depends on making connections between knowledge and views on the one hand and events that happen in the classroom on the other hand. Thus, teachers' expertise may also become apparent in the way they link their PCK on different levels of globality and use it as a lens for their noticing. This suggests that experienced expert teachers may distinguish themselves from novices by stronger interrelations and more consistency of their pedagogical content knowledge and views on the one hand, and their noticing on the other. Including pre-service as well as in-service teachers, the research presented here may thus contribute to identifying such differences between expert and novice teachers and to verifying corresponding assumptions within the scope of the study empirically.

While there is a broad basis of empirical research emphasizing the essential and delicate role of multiple representations for students' learning, research into corresponding professional knowledge and views of teachers is rather scarce. Investigations of teachers' PCK regarding representations concentrate so far largely on selection of representations and revolve around questions such as "Which representation does foster students' understanding best?" and "Which advantages and disadvantages do certain representations have?" (cf. e.g., Ball et al., 2008; Kunter, Baumert, Blum, Klusmann, Krauss, & Neubrand, 2011). However, in the light of the above reasoning this does not capture the full spectrum of that which comprises teachers' professional knowledge about dealing with multiple representations in the mathematics classroom. In particular, the key role of changing representations for students' understanding should not be neglected. To our knowledge there are merely a few studies regarding selected aspects of PCK about dealing with multiple representations in the sense of balancing the benefits and obstacles for understanding mathematics which are involved in using multiple representations. A qualitative study by Bossé, Adu-Gyamfi, and Cheetham (2011) has for instance focused on teachers' expectations of students being able to perform different conversions of representations. As using multiple representations and changing between them can foster as well as hinder students' learning depending delicately on different situation-specific aspects, teachers' noticing is very important for them to enable learners to benefit from dealing with multiple representation. However, there is a lack of research regarding such theme-specific noticing and also of studies taking the perspectives of teachers' professional knowledge and views together with evidence of their noticing.

Research questions

According to the need for research pointed out in the previous section, the study presented here aims to provide evidence for the following research questions:

1. What pedagogical content knowledge and views about the role of multiple representations for learning mathematics do (German) pre-service and in-service teachers have? In particular:
   (a) Which (global) reasons for using multiple representations in the mathematics classroom are most important to them?
   (b) What (content domain-specific) views about using multiple representations for teaching fractions do they have?
2. Do the teachers' evaluations of specific classroom situations indicate theme-specific noticing, that is, do they notice conversions of representations and their potentially hindering role for students' understanding?
3. Do in-service teachers and pre-service teachers differ with respect to such PCK and theme-specific noticing?
4. Is the teachers' theme-specific noticing interrelated with their corresponding pedagogical content knowledge and views? Is there evidence of stronger interrelations for in-service than for pre-service teachers?

5. Which components of professional knowledge are used for teachers' theme-specific noticing? Is it in particular possible to identify the use of knowledge on different levels of globality?

6. What content knowledge about dealing with multiple representations in the domain of fractions do pre- and in-service teachers have? Is this CK interrelated with their theme-specific noticing?

Sample and methods

For answering these research questions, a corresponding questionnaire was designed. At the beginning of this questionnaire, explanations of the notions *representation* and *pictorial representation* in a mathematical context were given in order to ensure a similar understanding of all participants regarding these central terms for the study.

The questionnaire was answered by a sample of German academic-track secondary school teachers: 67 pre-service teachers (33 female, 34 male) and 77 in-service teachers (35 female, 39 male, 3 without data). The pre-service teachers had a mean age of 21.4 years ($SD = 2.2$) and their average semester of teacher education at the university was 3.3 ($SD = 0.97$). The in-service teachers were on average 40.6 years old ($SD = 11.8$) and their mean teaching experience for mathematics was 12.4 years ($SD = 11.5$). The pre-service teachers answered the questionnaire at the beginning of a course at their university and the in-service teachers answered it at their schools in the presence of the first author or a student research assistant. Participants could take as much time as they needed to fill out the questionnaire.

Corresponding to our research questions, four sections of the questionnaire were considered in this study: Two sections were designed to assess pedagogical content knowledge and views about the role of multiple representations for learning mathematics on different levels of globality. A third section used a vignette-based format and was designed to elicit teachers' theme-specific noticing and the fourth section addressed the participants' specific CK, namely their ability to match symbolic-numerical representations of fractions with appropriate pictorial and content-related representations. In the following, these four sections are described in more detail.

In order to explore to what extent the pre-service and in-service teachers acknowledged the significance of multiple representations for mathematical understanding in general, the participants were asked to rate the importance of possible reasons for using multiple representations on a five-point Likert scale (from *not important* to *extremely important*). Some of these pointed to the special role that multiple representations play for building up conceptual mathematical knowledge, whereas the others expressed additional reasons, not specific to the subject of mathematics, such as addressing different learning types and input channels of the learners. The items for the second kind of reasoning were designed on the basis of a pilot study in which the participants could give reasons for using multiple representations in the mathematics classroom in an open format. Scales and sample items are presented in Table 5.5.

In order to capture domain-specific PCK on whether multiple representations should be emphasized when teaching fractions, a second questionnaire part comprising of multiple choice items was designed. For each of these items the participants could express their approval or disagreement on a four-point Likert scale (for details and sample items see Table 5.6).

For both of these questionnaire sections confirmatory factor analyses (CFA) were conducted in order to ensure the underlying structure of the intended constructs. The results of these analyses will be reported at the beginning of the results section.

For assessing the teachers' theme-specific noticing this study used a vignette-based format: The participants were given the transcripts of four fictitious classroom situations. We chose vignettes in the form of transcripts instead of videos since they better afford controlling for unintended disturbing factors such as external characteristics of the teacher and they make

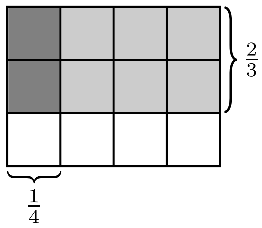
Scale (identifier)	Sample item	# items
Supporting remembering	Students can use pictorial representations as mnemonics.	3
Motivation and interest	They make it easier to keep students' interest.	3
Learning types and input channels	Different learning types and input channels can be addressed.	3
Necessity for mathematical understanding	Enhancing the ability to change from one representation to another is essential for the development of mathematical understanding.	4

Table 5.5: Scales regarding reasons for using multiple representations in the mathematics classroom

Construct (identifier)	Sample item	# items
Multiple representations (MR) for understanding	To understand fractions properly, it is necessary to use many different representations in class.	3
Multiple representations (MR) for individual preferences	In order to give students the opportunity to choose their preferred type of representation, which they most easily understand, they should be provided with many different representations.	3
One standard representation	It is best to use only one kind of pictorial representation for fractions in lessons, so that you can always come back to this as a 'standard' representation.	3
Fear of confusion by multiple representations (MR)	Several different pictorial representations for fractions could confuse students, especially the weaker ones.	3
Multiple representations (MR) impede learning rules	If students pay too much attention to pictorial representations, their ability to confidently do calculations with fractions is impeded.	3

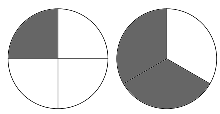
Table 5.6: Scales regarding views on dealing with multiple representations for teaching fractions

T illustrates the calculation $\frac{1}{4} \times \frac{2}{3}$ on the board:

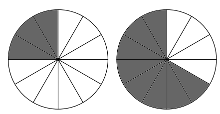


S: And how can you see here what $\frac{1}{4} + \frac{2}{3}$ is?

T: Well, this cannot be seen very well in this picture. For this it would be better to look at pizzas [draws]:



Before we can add the fractions, we have to make all the pieces the same size. Therefore we have to subdivide the pizzas:



Now we see that we have $\frac{3}{12}$ and $\frac{8}{12}$. So, if we add, we get $\frac{3+8}{12} = \frac{11}{12}$.

Figure 5.11: Sample vignette # 1 (T: teacher, S: student)

it therefore easier to design a coherent test instrument. All four classroom situations have in common that a student makes a comment revealing a misconception or asks a question and thus prompts the teacher to react somehow. In all four cases, the reactions by the fictitious teachers involve a change of representations: His or her answer uses another representation than the student without making explicit connections between the given and the newly introduced representation. Figures 5.11 and 5.12 show examples of such fictitious classroom situations.

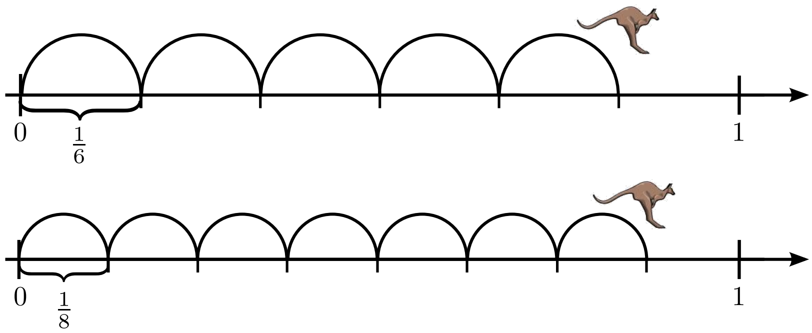
In the first case a student wants to know how you can see the addition of two fractions in the given rectangle, but the teacher explains the calculation using a pizza representation. No connections between the two representations are made. In fact, here it is easier to show the addition with the rectangle, since the subdivision into twelfths is already available. Hence, it is neither necessary nor advisable to force the student to engage with another representation at this point, even if previously the pizza representation had been used for adding fractions in this class.

In the case shown in Figure 5.12, a student reveals a misconception by claiming that the fractions $\frac{5}{6}$ and $\frac{7}{8}$ were of equal size, since "there is always one piece of the whole missing". As a reaction the teacher draws two number lines with kangaroos jumping five times $\frac{1}{6}$, compared with seven times $\frac{1}{8}$ starting at zero and argues that the first kangaroo does not get as far as the second one. Hence, a new representation is used and it is not appropriately connected with the student's verbal representation when speaking of missing pieces of the whole.

We used four such items to assess the participants' theme-specific noticing. All of these items have in common that a change of representations is conducted that is potentially hindering for students' understanding and not necessary from the content point of view. The teachers were asked the following question with respect to all four classroom situations in

S: The fractions $\frac{5}{6}$ and $\frac{7}{8}$ are of equal size! There is always one piece of the whole missing.

T: Hmm, let's have a look at this on the number line: [draws]



Now you can see that the kangaroo that jumps five times $\frac{1}{6}$ does not get as far as the kangaroo that jumps seven times $\frac{1}{8}$, can't you?

Figure 5.12: Sample vignette # 2 (T: teacher, S: student)

this questionnaire section: "How much does this response help the student? Please evaluate the use of representations in this situation and give reasons for your answer". Prompting the participants to evaluate the use of representations instead of asking them to evaluate the teacher's reaction more generally has the advantage that they had a more focused opportunity to show theme-specific noticing in their answers, since otherwise not having enough time or space to list all possibly interesting aspects of the teacher's reaction would have been more likely to prevent their answers from indicating their theme-specific noticing. A similar methodological approach has also been used in the question format of the study by Jacobs et al. (2010) on professional noticing of children's mathematical thinking. In view of the corresponding research question, the teachers' answers were analyzed concerning two main aspects: paying attention to the change of representations and sensitivity to potentially negative effects of this change of representations for the students' understanding. Hence, in a first step the answers of the participants were coded with respect to the following two guiding questions: "How was the teacher's response evaluated?" and "Which role does the teacher's use of representations play in the justification for this evaluation?" The categories for the corresponding top-down coding related to these two aspects and corresponding sample answers (referring to the sample item in Figure 5.11) are listed in Table 5.7.

All answers were double-coded by the first author and a student research assistant with high interrater reliability: Cohen's kappa was in both cases .91. Discrepancies were resolved through discussion. Subsequently, in a second step the answers receiving the combination of the codes *negative evaluation* or *balanced evaluation* regarding the first question and *justification referring to the change of representations* for the second question were considered to indicate the theme-specific noticing that this study targets. This approach yields a simple measure for such noticing: Each participant receives a score for their theme-specific noticing that counts in how many (out of four) cases his or her answer indicates that the change of representations and its critical role for the student's understanding was noticed.

The last questionnaire section which is included in this study focuses on specific CK about dealing with multiple representations in the content domain of fractions. More specifically, this CK test was designed to assess the participants' ability to match symbolic-numerical representations of fractions and their operations with appropriate pictorial and content-related representations. As in the sample item shown in Figure 5.13, given (incorrect) conversions between such representations had to be checked and corrected or a conversion had to be carried out.

How was the teacher's response evaluated?		
Code		Sample answer
0	No evaluation	
1	Positive evaluation	"I find the reaction acceptable and good. Especially f or adding fractions, pizzas are still most suitable."
2	Balanced evaluation	"For many students this is too difficult, although the addition is well explained in the example."
3	Negative evaluation	"Not good. The question was not answered."
Which role does the use of representations play in the justification for this evaluation?		
Code		Sample answer
0	No justification given for the evaluation	
1	Justification without referring to representations	"The student is confused, since multiplication suddenly turns into addition."
2	Justification referring to representation(s), but not to the change of representations	"Especially for adding fractions pizzas are still most suitable."
3	Justification referring to the change of representations	"T. could and should have shown the addition using the partitioned rectangle. The change hardly helps the student."

Table 5.7: Categories and sample answers (answers related to item in Figure 5.11)

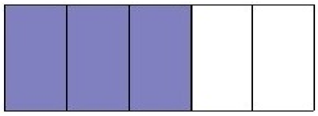
Please change the diagram, if necessary, so that $\frac{3}{5}$ of $\frac{1}{4}$ is shaded. Otherwise just tick the box on the right-hand side.		

Figure 5.13: Sample item of the CK test

Results

The teachers' pedagogical content knowledge and views about the role of multiple representations for learning mathematics

Focusing on the first research question we start with the results concerning the two questionnaire parts assessing pedagogical content knowledge and views about the key role of multiple representations for learning mathematics. In the first instance we take a look at the teachers' evaluations of possible reasons for using multiple representations. In order to control the theoretically presumed structure of the corresponding questionnaire instrument (cf. Table 5.5), a confirmatory factor analysis was conducted on the items in this section. The appropriateness of our theory-based model was assessed by several measures of global model fit. The *RMSEA* (root mean square error of approximation) can be interpreted as the amount of information within the empirical covariance matrix that cannot be explained by the proposed model. The model may be classified as acceptable if at most 8 % of the information is not accounted for by the model, that is, $RMSEA \leq 0.08$ (Kline, 2005). The current model meets this criterion as *RMSEA* is 0.056. Furthermore, measures of incremental fit were employed, namely the *TLI* (Tucker-Lewis index) and the CFI (comparative fit index). An acceptable fit is in both cases indicated by values ≥ 0.90 (Kline, 2005). For the proposed model the *TLI* is 0.93 and the *CFI* is 0.94 and hence these criteria are met as well. Thus, although the χ^2 statistic is significant ($\chi^2 = 86$, $DF = 59$, $p = .013$), it is concluded that the model fits the data reasonably well. As regards the local model fit, the analysis yielded that all factor loadings are highly significant ($p < .001$). Hence, even though the factor reliabilities range from .62 to .76, which is (probably due to the small number of items) not very high, but still acceptable, for each of the four constructs a corresponding scale could be formed. Figure 5.14 shows the means and their standard errors of these four scales for the pre-service and in-service teachers separately.

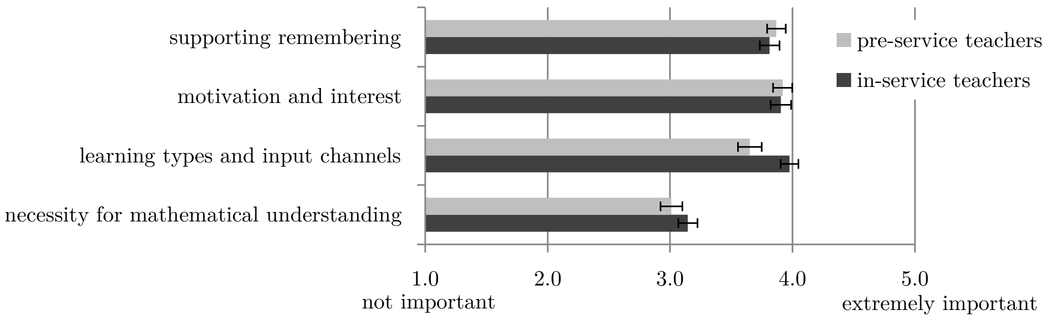


Figure 5.14: Views related to reasons for using multiple representations (means and their standard errors)

Remarkable about the results presented in Figure 5.14 is first of all that the pre-service as well as the in-service teachers saw on average the special role of multiple representations for understanding mathematics as less important than the other reasons. Comparing the means of the two subsamples for each scale shows little differences. Only for the scale "learning types and input channels" there is a significant difference: The in-service teachers in our sample attached a higher importance to such reasons than did the pre-service teachers ($t(141) = 2.0$, $p < .01$, $d = 0.46$).

Addressing next the teachers' domain-specific PCK on whether multiple representations should be emphasized when teaching fractions, the appropriateness of the theory-based structure of the corresponding questionnaire section (cf. Table 5.6) was checked in a first step: conducting a confirmatory factor analysis, the model exhibited a reasonably good data fit ($RMSEA = 0.05$, $TLI = 0.96$, $CFI = 0.97$). Since all the factor loadings are highly significant and all five factors are reasonably reliable with α ranging from .74 to .82, corresponding scales could be formed.

The comparison of the two subsamples regarding their means for these scales, which is presented in Figure 5.15, shows clear differences. Hence, in contrast to the global views focused on before, domain-specific views about using multiple representations for teaching fractions were distinct for the pre-service and in-service teachers. The in-service teachers were more in favor of using multiple representations for teaching fractions than the pre-service teachers. The effect sizes of these differences which were measured by Cohen's d are medium to large. Judging from these effect sizes, the biggest differences between the participating pre-service and inservice teachers was detected by the scale "Multiple representations (of fractions) for understanding".

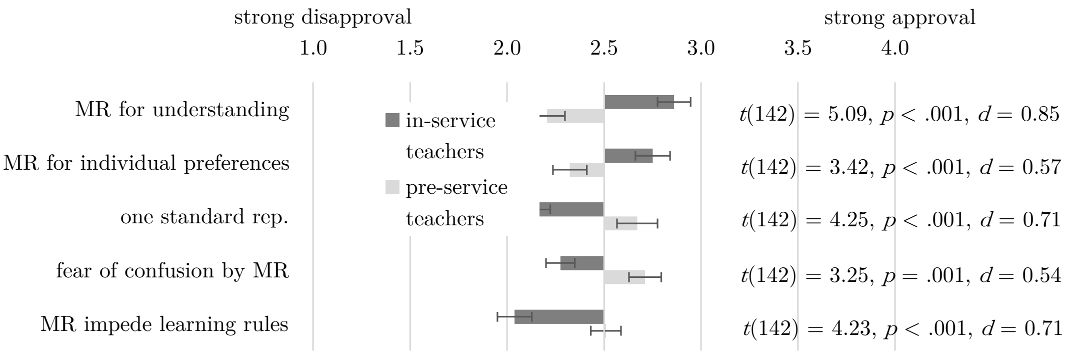


Figure 5.15: Views on the role of multiple representations (MR) for teaching fractions (means and their standard errors)

The teachers' theme-specific noticing and interrelations with their corresponding PCK

Addressing our second research question, we focus next on the participants' theme-specific noticing: Figure 5.16 shows the mean scores and their standard errors for the samples of pre-service and in-service teachers.

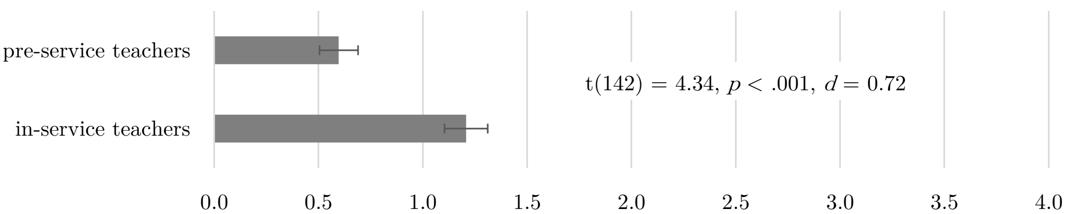


Figure 5.16: Mean number of answers (out of four) that indicate theme-specific noticing

On average, the participants showed a relatively low frequency of theme-specific noticing. However, the answers of the in-service teachers indicated such noticing on average about twice as often as those of the pre-service teachers. This difference is highly significant and represents a medium effect (cf. Figure 5.16).

In order to find answers to the fourth research question of this study, which concerns interrelations between teachers' theme-specific noticing and their corresponding professional knowledge and views, we examine in a first approach whether the score for theme-specific noticing correlates with scales assessing relevant views. Since the theme-specific noticing which was addressed by this study focuses on conversions of representations and their evaluation, it is likely that teachers draw on their pedagogical content knowledge and views about the role of conversions of representations for the students' understanding. The scales "necessity for mathematical understanding" (on the global level) and "multiple representations (of fractions) for mathematical understanding" (on the content domain-specific level) reflect such professional knowledge and views, as can be seen by reviewing the corresponding sample items presented in Table 5.5 and in Table 5.6 – they emphasize the crucial role of conversions of representations for students' understanding. Consequently we examined whether these two scales correlate with the score for theme-specific noticing.

Considering Pearson's correlation coefficients in Table 5.8 shows that no correlations could be found in the case of the pre-service teachers. In contrast, the in-service teachers' theme-specific noticing was significantly related to the global view that the ability to change between representations is essential for the development of mathematical understanding (medium effect). The relationship between theme-specific noticing and the domain-specific view on multiple representations of fractions was not significant.

		General reason: "necessity for mathematical understanding"	"MR (of fractions) for mathematical understanding"
Theme-specific noticing	Pre-service teachers	−.02 ns	−.01 ns
	In-service teachers	.32**	.18 *ns*

ns = not significant ($p \geq .05$), * $p < .05$, ** $p < .01$

Table 5.8: Pearson's correlation coefficients for the score "theme-specific noticing"

Qualitative analysis of selected cases on teachers' answers

Complementing this quantitative approach by a qualitative in-depth analysis and addressing the fifth research question of how the teachers' theme-specific noticing was informed and shaped by different components of their professional knowledge, we focus now on selected cases of teachers' answers. Figures 5.17, 5.18, 5.19, 5.20, 5.21 and 5.22 show examples of teachers' answers to the first sample item (cf. Figure 5.11) which indicate successful theme-specific noticing (originals and their English translations). In the following we will analyze them with regard to which components of professional knowledge and views this noticing draws on and in particular we will focus on the question as to whether these professional knowledge components can be allocated on different levels of globality (cf. model in Figure 5.10).

Da die Frage von S sich auf die Darstellung mit dem Rechteck bezog, hätte L bei dieser Darstellung bleiben sollen. Er wäre so dem Interesse von S mehr gerecht geworden. Außerdem hätte das erneute Zeichnen einer Rechteckfigur keinesfalls mehr Zeit gekostet als das Zeichnen der Kreisfigur.	Since the question by S refers to the representation with the rectangle, T should have stayed with this representation. Like this he would have done justice to the student's interest. Moreover, drawing a rectangle again would certainly not have cost more time than drawing the circle.

Figure 5.17: Mr. A's answer (in-service teacher)

Mr. A (cf. Figure 5.17) argues that connecting to the student's question which refers to the rectangle the teacher should have answered by using this representation. We may conclude indirectly from this statement that Mr. A probably knows that it is not necessary to change the representation in this situation and he adds that drawing the circle does not even save time. For Mr. A it obviously is important to do justice to the student's interest in this situation. Hence, he analyzes the teachers' reaction using *knowledge which is closely tied to aspects of the specific instructional situation*: the students' particular interest and the effort of drawing the particular representations in this situation.

Die Addition bzw Summe kann am ersten Bild genauso gut gezeigt werden, da die Zwölftel Unterteilung bereits da ist. Es muß eben anders angeordnet werden, das Ergebnis $\frac{11}{12}$ ist sofort einsichtig! Es stört auch, wenn auf 2 Ganze übergegangen wird. Das ist für Schüler ein neues Problem.	The addition resp. the sum can as well be shown using the first picture, since the subdivision into twelfths is already there. It just has to be arranged differently, the solution $\frac{11}{12}$ is immediately clear! It is also disturbing to pass on to 2 wholes. This is a new problem for students.

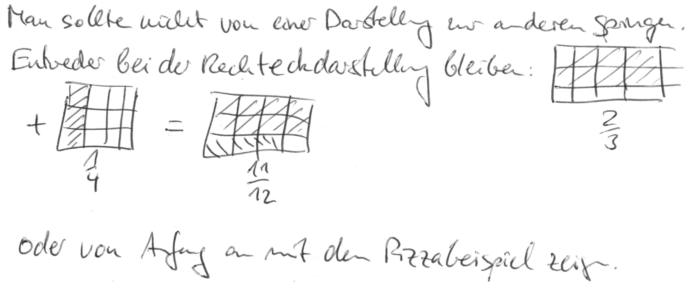
Figure 5.18: Ms. B's answer (in-service teacher)

Ms. B (cf. Figure 5.18) reasons in her answer about the advantages of the rectangle representation for adding fractions and about problems that students may have with passing on from a representation with one whole to another with two. This reasoning may therefore indicate that her theme-specific noticing was mainly informed by her *knowledge related to the particular content.*

Sie verwirrt den Schüler mehr anstatt ihm zu helfen. Dadurch wird das Verständnis für einen Zusammenhang zwischen Multiplikation und Addition verschlechtert, da unterschiedliche Darstellungen benutzt werden. Addition und Multiplikation sollten in den gleichen Darstellung erklärt werden.	It [the reaction] confuses the student more instead of helping him. As a result the understanding of a connection between multiplication and addition gets worse, since different representations are used. Addition and multiplication should be explained in the same representation.

Figure 5.19: Ms. C's answer (pre-service teacher)

Ms. C notes that the teachers' reaction is confusing rather than helpful for the student and she identifies the use of different representations regarding multiplication and addition as being problematic: She argues that explaining addition and multiplication (of fractions) should be done by using the same representation. Compared to Ms. B her argumentation is on a more general level, since she does not refer to the specific representations; Ms. C evaluates the teachers' reaction by reasoning on the base of on her *domain-specific knowledge and views.*

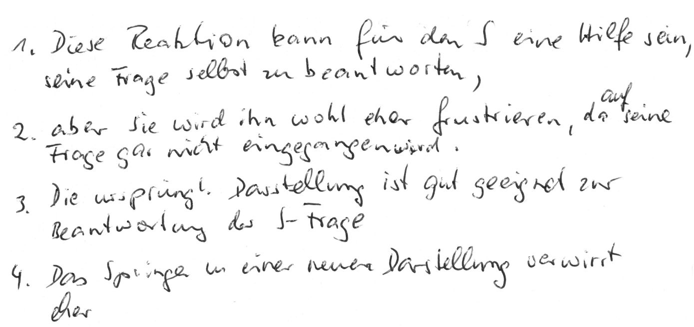


One should not jump from one representation to the other. Either stay with the rectangle representation: [drawing] Or use the pizza example from the beginning.

Figure 5.20: Mr. D's answer (in-service teacher)

Mr. D starts his evaluation with the very general assertion that "one should not jump from one representation to the other" and then he applies this global component of his professional knowledge to the specific classroom situation. Hence, he may above all have used *professional knowledge and views that are not tied to a specific content domain* as a lens for his noticing.

While in these cases the teachers' theme-specific noticing was apparently informed mainly by professional knowledge on one level of globality respectively, there are other answers which suggest that the teacher has drawn on and combined different levels of corresponding PCK.



1. This reaction can help S to answer his question himself,
2. but it will probably rather frustrate him, since there is no response to his question.
3. The initial representation is well suited for answering P's question.
4. Jumping to a new representation is rather confusing.

Figure 5.21: Mr. E's answer (in-service teacher)

The answer of Mr. E (cf. Figure 5.21) is such an example. Similar to Mr. A's answer, his first two points refer to the particular student and his question in this specific instructional situation and thus they indicate *situation-specific knowledge and views.* Next, Mr. E points out that the initial representation is appropriate for answering the student's question and hence he draws on *knowledge about the particular content*, namely about the rectangle representation for fractions. In his last point he states that jumping to a new representation is rather confusing. Even though it is possible that he was thinking of the specific change of representation when writing this, it is noteworthy to mention that Mr. E's assertion is phrased in a relatively general way and therefore it probably expresses *general professional* *knowledge* about changing representations and their role for students' understanding. Mr. E's answer indicates moreover a high consistency between different professional knowledge components and his theme-specific noticing. He appears to be able to generalize from his situation-specific observation and to make connections to his more general knowledge.

Konkret auf die Frage „wie kann man da sehen" hilft die L-Antwort dem S nicht weiter. Sie hilft ihm auch nicht dabei weiter, flexibel die verschiedenen Darstellungsarten zu benutzen, sondern fördern die Fehlvorstellung, dass Rechtecke für Multiplikationen und Kreise für Additionen da sind.

Die versch. Darstellungen sollen nicht verschiedenen Rechenoperationen zugeordnet werden, sondern vielmehr sollten Aufgaben auch mehrfach – mit verschiedenen Darstellungswegen – gelöst werden. Denn der flexible Umgang mit den versch. Darstellungen fördert das Verständnis.

With respect to the question "how can you see here", the answer by T does not help S. It also does not help him to use the different forms of representations flexibly, but instead it encourages the misconception that rectangles are for multiplication and circles for addition. The different representations should not be assigned to different calculations, but problems should rather be solved multiple times with different ways of representations. Since dealing flexibly with the different representations fosters the understanding.

Figure 5.22: Mr. F's answer (in-service teacher)

Similarly, Mr. F's evaluation (cf. Figure 5.22) draws on professional knowledge on different levels of globality: First he argues that the teacher's answer is not appropriate for the student's question and therefore not helpful in this situation. Making this point, Mr. F uses *knowledge which is tied to the specific instructional situation.* Then he adds that the teacher's reaction does not encourage flexible use of multiple representations, but the misconception that different representations must be used for adding and multiplying fractions. He asserts that the same problems should rather be solved repeatedly using different representations. For this argumentation Mr. F refers to how multiple representations should be used for fraction calculations, and thus he may draw on *content domain-specific knowledge*, but it also involves more general parts, where he appears to have in mind goals about students' competencies regarding representations which are not specific to a particular content domain. Especially in his concluding remark ("dealing flexibly with multiple representations fosters the understanding") his words indicate that his analysis of the given student-teacher interaction is also informed by *global professional knowledge* about the role of multiple representations for learning mathematics. Similarly to Mr. E's answer, Mr. F's evaluation suggests strong interrelations between his specific components of professional knowledge on different levels of globality and his theme-specific noticing.

The analysis of these examples of participants' answers shows that their successful theme-specific noticing was informed by a variety of different components of their corresponding professional knowledge and views. It was found that these components may stem from all four different levels of globality according to the model of components of professional knowledge (cf. Figure 5.10). Moreover, theme-specific noticing can on the one hand be successful when drawing on mainly a single level of globality versus situatedness, but on the other hand it may also involve and combine components of professional knowledge on several levels.

Cases of participants' answers, which do not indicate that they have noticed the change of representations and that it should be seen as critical, can also be explored regarding how these evaluations were informed by professional knowledge and views and they may make it possible to identify possible hindering factors for theme-specific noticing.

Die Reaktion hilft dem Schüler gut weiter. Da man Bruchrechnen mit Pizzas ganz leicht darstellen kann.	The reaction helps the student well. Since fraction calculations can be represented by pizzas very easily.

Figure 5.23: Mr. G's answer (pre-service teacher)

Mr. G (cf. Figure 5.23) for instance approves of the teacher's reaction and he justifies this evaluation by expressing his view that pizzas are a good way of representing fraction calculations. This answer appears to draw on a domain-specific pedagogical content view. It may not be wrong, but the problem is that he merely attends to the teacher's use of the pizza representation and not to the change of representations. For this reason his answer was not coded to indicate theme-specific noticing. The evidence does not show whether other corresponding professional knowledge could have helped Mr. G to show theme-specific noticing. However, if he has such professional knowledge about the role of changing representations for students' understanding, it appears to have been suppressed by his view about the pizza representation which he draws on for analyzing the teacher's reaction.

Ich denke eine klare Trennung von Multiplikation & Addition ist hier sinnvoll. Der S merkt so, dass er beim Addieren anders vorgehen muss (er denkt an die Pizza) als beim Multiplizieren (Quadrate) Dabei eine andere Form der Darstellung zu wählen ist deshalb denke ich sinnvoll.	I think a clear separation between multiplication and addition is reasonable here. In this way S realizes that he must proceed differently for adding (he thinks of the pizza) than for multiplying (squares). Choosing a different form of representation is thus reasonable, I think.

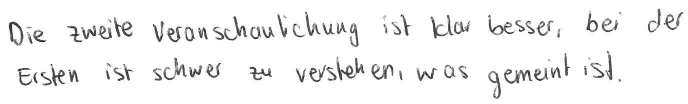
Figure 5.24: Ms. H's answer (in-service teacher)

The case of Ms. H (cf. Figure 5.24) shows in turn that recognizing the teacher's change of representations is not sufficient for theme-specific noticing. In contrast to Mr. F she argues that the teacher's choice of a different representation in the sense of a clear separation of addition and multiplication is reasonable for the student's learning. She hence draws predominantly on her pedagogical content views on how to teach addition and multiplication of fractions to evaluate the teachers' reaction, which makes her approve. These views appear to prevent her from seeing the potentially hindering role of the conversion of representations for the student's understanding.

Die Lehrkraft hätte sagen sollen, dass beim Bruchrechnen zwischen Addition und Multiplikation ein Unterschied besteht. Man kann beide Rechenarten nicht in einer selben Darstellung zeigen.	The teacher should have said that regarding fraction calculations, there is a difference between addition and multiplication. You cannot show both operations in the same representation.

Figure 5.25: Ms. J's answer (pre-service teacher)

The evaluation by Ms. J (cf. Figure 5.25) also indicates that she has not realized that the change of representations in the specific situation should be seen critically. Her answer appears to reveal rather a lack of specific content knowledge and questionable pedagogical content views: She claims that one cannot explain addition and multiplication of fractions using the same kind of representation. The absence of the necessary CK about representations of fraction operations apparently prevents her from an appropriate evaluation of the teacher's reaction and thus from theme-specific noticing.



The second visualization is clearly better, in case of the first one it is difficult to understand what is meant.

Figure 5.26: Mr. K's answer (pre-service teacher)

The case of Mr. K (cf. Figure 5.26) shows even more explicitly that lacking specific content knowledge can obstruct theme-specific noticing. He compares the two pictorial representations and claims that the pizza representation is clearly better, since "in case of the first one it is difficult to understand what is meant". He appears not to understand the rectangle representation for multiplication of fractions.

Considering all the cases in which theme-specific noticing of the participants was not successful, two main reasons could be identified. Either they have apparently not paid attention to the change of representations or they have recognized the change of representations, but then, by drawing on rather inappropriate views or even wrong knowledge, they did not arrive at the conclusion that it should be seen critically. In the first case one possibility is that other, more dominant, views which are connected to the classroom situation might prevail over focusing on the change of representations in the first place. Hence, in both cases unsuccessful theme-specific noticing often appears to be connected to the teachers' professional knowledge and views. The answer by Mr. G (cf. Figure 5.23) may serve as an example for the first case, whereas the evaluations by Ms. H (cf. Figure 5.24), Ms. J (cf. Figure 5.25) and Mr. K (cf. Figure 5.26) represent the second case.

The teachers' specific CK about dealing with multiple representations in the domain of fractions

In the light of the fact that some of the participants' answers have revealed a lack of specific content knowledge – which was apparently obstructing theme-specific noticing – we next address research question 6 and focus on the results regarding the specific CK test included in the questionnaire. Figure 5.27 shows the mean scores (and their standard errors) of both subsamples for this test. It is apparent that the pre-service teachers achieved significantly lower scores than the in-service teachers. In order to explore whether the on average lower specific content knowledge of the pre-service teachers in our study may have been a hindering factor for their theme-specific noticing, we examined correlations between the score for theme-specific noticing and the score for specific content knowledge.

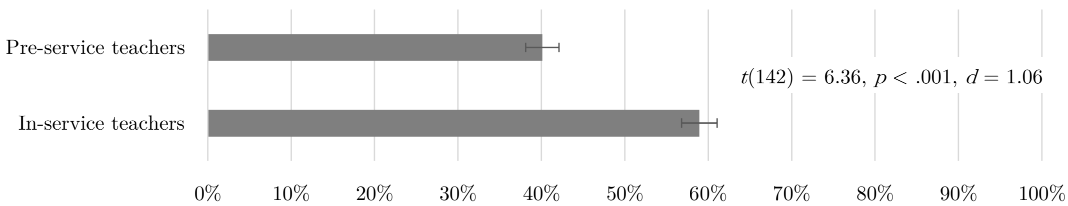


Figure 5.27: Specific content knowledge scores (means and their standard errors)

Analyzing the corresponding correlation coefficients (Pearson) for both subsamples separately shows that these two scores were weakly but significantly related with respect to the pre-service teachers ($r = .25$, $p < .05$), whereas there was no correlation found for the sample of in-service teachers ($r = .03$, *not significant*).

Discussion and conclusions

The results of this study may serve three purposes: Firstly they can give an insight into what kind of knowledge and views teachers have about the role of multiple representations for learning mathematics and into their corresponding theme-specific noticing in the sense of identifying prerequisites and specific needs for teacher professional development. Secondly, they can contribute to a better understanding of how such noticing is informed and shaped by different components of professional knowledge and views. Thirdly, the comparison of pre-service and in-service teachers allows us to identify differences between expert and novice teachers and review corresponding assumptions within the scope of this study empirically. Before these results are discussed in more detail we would however like to emphasize the limitations of this study which suggest interpreting the evidence with care: The study is not representative of German pre-service or in-service teachers. Furthermore, even though a spectrum of different facets of professional knowledge about the role of multiple representations for students' understanding were taken into account for the design of the study, the constructs can merely give an indicator-like insight into some aspects of such teacher knowledge and views. The qualitative in-depth analysis of instances of teacher noticing has shown a broad variety of views and knowledge components that have played a role for the teachers' theme-specific noticing. These aspects point to more facets than we could assess as constructs in the quantitative part of this study. Bearing this in mind, the findings however allow answering the research questions and indicate several aspects of theoretical and practical relevance.

Regarding general views and knowledge about the role of multiple representations for learning mathematics, the findings of this study suggest a similar profile for pre-service and in-service teachers: Both subsamples have on average attached less significance to reasons for using multiple representations that reflect the special role for developing conceptual understanding of mathematics compared to other reasons that are rather not discipline-specific. This result may be seen in line with findings reported by Ball (1993a) that American teachers emphasized the motivational potential associated with (pictorial) representations and appeared to neglect their role for conceptual understanding. Both studies hence point to a lack of awareness of the fact that successful mathematical thinking usually depends on the interplay of different representations and hence the findings indicate a need for specific professional development.

Focusing on more situated views and knowledge regarding the role of multiple representations for teaching fractions has however yielded clear differences between the pre-service and in-service teachers who have taken part in our study. The in-service teachers have on average approved more of using multiple representations for teaching fractions and in particular they have attached greater significance to the role of changes of representations for the students' understanding than the pre-service teachers. Thus, at least when it comes to views and aspects of knowledge about dealing with multiple representations that are related to the specific content domain of fractions (and that are therefore more situated), the practicing teachers appear to distinguish themselves from the pre-service teachers in the sense of having more developed pedagogical content views.

Regarding the participants' theme-specific noticing, the results of this study yield further evidence of a need for specific teacher professional development. Overall, the teachers noticed relatively rarely conversions of representations and their potentially hindering role for students' understanding in examples of student-teacher interactions. However, the evaluations by in-service teachers indicated such noticing on average about twice as often as for pre-service teachers. This finding may be seen as quantitative evidence for noticing being a characteristic of expert teachers and thus it can add to corresponding results from previous studies (e.g., Berliner, 1994; Jacobs et al., 2010). Then again it shows that experienced teachers are not necessarily experts when it comes to theme-specific noticing.

For finding answers to the question of how teachers' theme-specific noticing is interrelated with and informed by their corresponding views and professional knowledge, we have used two different approaches. Firstly, such relations were analyzed by looking at possible correlations between the particular scores for each scale. Secondly, by an exploratory analysis of teachers' answers we could identify different components of professional knowledge and views that have informed the teachers' theme-specific noticing. With respect to the quantitative approach, the results of this study suggest that corresponding global and domain-specific views about the role of changing representations for students' understanding were only weakly related to the teachers' theme-specific noticing. However, the theme-specific noticing of the in-service teachers was significantly related to the global view that the ability to change between representations is essential for the development of mathematical understanding. This may point to comparatively stronger interrelations of the in-service teachers' pedagogical content knowledge and views on the one hand and their theme-specific noticing on the other hand. For the pre-service teachers in turn, the findings of this study suggest that specific content knowledge has played an important role for their theme-specific noticing. Their average score for specific CK was significantly lower compared to the in-service teachers. This may seem surprising in light of the fact that the pre-service teachers have more recently attended mathematics courses at university. However, matching symbolic-numerical representations of fractions and their operations with appropriate pictorial and content-related representations is a central element of teaching fractions and less a content of university lectures, which may be the reason why the in-service teachers scored on average higher than the pre-service teachers regarding this specific CK. The pre-service teachers' theme-specific noticing was weakly positively related to this score. Hence, their on average relatively low specific content knowledge was apparently a small but significant hindering factor for their theme-specific noticing and thus they may need a strengthened CK background. Teacher education should combine specific support in this field with learning opportunities connected to content-specific PCK and theme-specific noticing.

The fact that the correlations that were found between theme-specific noticing and the measured components of corresponding professional knowledge and views were merely weak to medium may be better understood against the background of the complementary qualitative findings. The analysis of cases has shown that there is not a simple relationship between successful theme-specific noticing and a single component of professional knowledge. Instead, drawing on a variety of different components of professional knowledge and views can result in successful theme-specific noticing. In particular it has become evident that such components can be allocated on a spectrum of different levels of globality: Situated knowledge and views closely tied to the instructional situation could serve the purpose as well as very global professional knowledge about changing representations, which is not specific to the content domain.

Furthermore, observations regarding cases in which theme-specific noticing was not indicated suggest that lack of professional knowledge, rather inappropriate views, as well as selective use of professional knowledge may be hindering factors. Concerning the latter case, it is possible to argue that teachers simply do not attend to the change of representations and thus no corresponding knowledge or views are drawn on. It is however also possible that other views or knowledge components are more dominant and thus prevail over focusing on the change of representations in the classroom situation (cf. Kuntze & Dreher, 2014).

In conclusion, teachers' professional knowledge about the role of multiple representations for learning mathematics has to be considered further as an important prerequisite for theme-specific noticing in student-teacher interactions. Thus the results of this study suggest that it is necessary to address the lack of understanding of the key role of multiple representations for mathematical understanding in a majority of the participating pre-service and in-service teachers by specific teacher professional development.

In order to gain more insight into the way in which teachers' noticing and their professional knowledge and views are interrelated, further research should focus as well on other themes and on different content domains. Moreover, the results of this study should be replicated by similar studies using a modified design of how (theme-specific) noticing is assessed, that is, by using more open question formats that do not prompt the participants to focus on a particular theme or by using classroom videos instead of text-vignettes for a more realistic presentation or simulation of a classroom situation. In view of the fact that aspects of professional knowledge and views as well as what teachers notice regarding the role of multiple representations for mathematical understanding may be culture-dependent (cf. Dreher et al., submitted; Miller & Zhou, 2007) future studies should also include teachers from different countries in order to identify such differences. A further follow-up question may be how teachers' professional knowledge and views about the role of multiple representations for mathematical understanding and their theme-specific noticing are related to their students' competencies in dealing with multiple representations. The results of a study from an ongoing project focusing on such questions could soon give some answers (cf. Dreher, Winkel, & Kuntze, 2012).

5.3 Article 3

Teachers facing the dilemma of multiple representations being aid and obstacle for learning – Evaluations of tasks and theme-specific noticing

Anika Dreher, Sebastian Kuntze

Abstract
Using multiple representations plays a double role for learning mathematics: On the one hand changing between representations is essential for mathematical understanding, but on the other hand such changes can involve excessive demands and thus hinder learning. Balancing this dilemma appears consequently to be important for successfully teaching mathematics. Despite such significance it is however little known how teachers take into account this phenomenon when they select tasks for the mathematics classroom or whether they notice the occurrence of corresponding obstacles in student-teacher interactions. Therefore, this study focuses on teachers' evaluations of the learning potential of tasks which make use of multiple representations in different ways and on their so-called theme-specific noticing. Since the teachers' views on how to deal with the dilemma may depend on whether they address higher- or lower-achieving students, more than 100 German mathematics teachers from two different secondary school types ("Gymnasium" and "Haupt-/Werkrealschule") were included in the study. The results suggest generally a rather low awareness of the double role of multiple representations for students' learning, but they also indicate significant differences between the two subsamples in the way they take account of the two sides of the dilemma in situated contexts.

Keywords: Multiple representations, Evaluation of tasks, Theme-specific noticing, Teachers' views, Fractions

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Introduction

Although there is an ongoing discussion about whether teacher expertise consists in dealing with the dilemmas associated with teaching in a reflective way (cf. Baumert & Kunter, 2006; Helsper, 2007), it is widely acknowledged that teachers often have to balance conflicting requirements (e.g., Ball, 1993b; Lampert, 1985; Syring, Bohl, Kleinknecht, Kuntze, & Rehm, in press). From a content-specific point of view in mathematics education using multiple representations in the mathematics classroom involves such conflicting requirements: On the one hand fostering the construction processes of students' conceptual understanding by using multiple representations and on the other hand protecting students from the excessive cognitive demands which are often involved in changing between representations (e.g., Ainsworth 2006; Duval, 2006). Balancing the corresponding dilemma does not only mean to be generally aware of both sides in the sense of holding corresponding views. Even more importantly it should also become evident in teachers' reflection related to situated contexts, for instance regarding the teachers' evaluation of tasks or their noticing of significant events in the classroom. Against this background a study was designed which focuses on teachers' evaluations of the learning potential of tasks and on their noticing – through the theoretical lens of dealing with the dilemma of multiple representations as being aid and obstacle for learning. As far as the mathematical content is concerned, the study is set in the domain of fractions, since using multiple representations is particularly relevant in this content domain. Taking into account mathematics teachers from different secondary school types (higher-achieving and lower-achieving students) allows moreover to explore whether these teachers differ in the way they deal with the dilemma associated with using multiple representations.

Theoretical background

The double role of multiple representations for learning mathematics

National standards in many countries emphasize the importance of dealing with multiple representations for learning mathematics (e.g., KMK, 2003; NCTM, 2000). In the German standards for the mathematics classroom, for instance, "using mathematical representations" is stated as one out of six general aspects of mathematical competence. It includes "applying, interpreting, and distinguishing different representations for mathematical objects and situations", "recognizing connections between representations" and "choosing different representations depending on the situation and purpose and changing between them" (cf. KMK, 2003, p 8, translation by the authors). There are very good reasons for such an emphasis of representations in the mathematics classroom: Representations play a major role in all kinds of mathematical activities, since mathematical objects are not accessible without them (Duval, 2006). Figure 5.28 shows an example which illustrates this phenomenon: Some possible representations for the fraction $\frac{3}{4}$ are given, but none of them is the fraction itself. They can merely stand for the mathematical object and make visible different aspects and characteristics of it.

Accordingly, we take the notion *representation* to mean something which stands for something else – in this case for an 'invisible' mathematical object (cf. Duval, 2006; Goldin & Shteingold, 2001). In order to see different facets of the corresponding mathematical object and for developing an appropriate concept image, usually several different representations have to be integrated (e.g., Ainsworth, 2006; Duval, 2006; Even, 1998; Goldin & Shteingold, 2001; Tall, 1988). Hence, representing mathematical objects in multiple ways plays an essential role for mathematical understanding (Duval, 2006; Elia et al., 2007; Even, 1998). However, a representation does not stand for a mathematical object in any obvious, self-explanatory way. Instead, this connection is subject to interpretation and negotiation processes (Gravemeijer et al., 2002; Meira, 1998) and it can only be created in the interaction

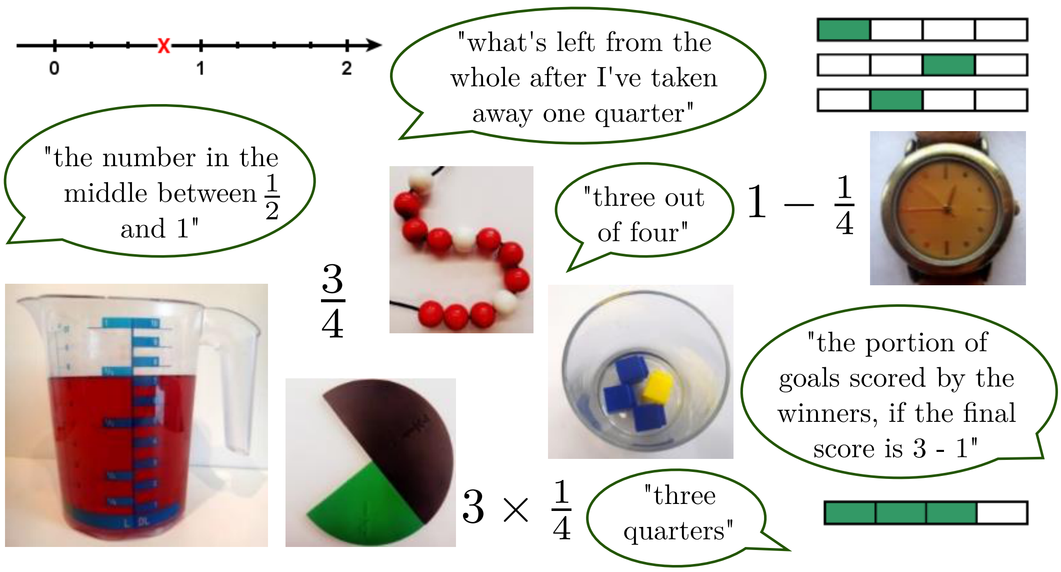


Figure 5.28: Some representations for the fraction $\frac{3}{4}$

between the participants in a learning environment (Steinbring, 2000). The so-called onto-semiotic approach emphasizes moreover that even the mathematical objects only emerge from the actions and discourse through which they are expressed and communicated (Font et al., 2013). Consequently, for each mathematical representation they are faced with, students have to construct meaning and to learn how it is used and interpreted in their classroom and in the discipline. So as not to confuse the representation with the mathematical object it stands for, it is however important not to deal with the representation in an isolated way, but to make connections with other representations of the corresponding object (Duval, 2006). This affords in turn going beyond the specific representation and being able to change between different representations (Even, 1998; Kaput, 1989). However, these cognitive processes are often highly demanding for learners and can become an obstacle to understanding (Ainsworth, 2006; Duval, 2006; Sfard, 2000). Multiple representations hence play a double role for learning mathematics: On the one hand they are an essential aid for building up conceptual mathematical understanding and the ability to deal with them flexibly is key to successful mathematical thinking and problem solving (e.g., Acevedo Nistal et al., 2009; Lesh et al., 1987; Stern, 2002; Zbiek et al., 2007). On the other hand multiple representations can hinder learning mathematics, since interpreting them, recognizing their connections and changing between them are challenging tasks – and yet necessary for benefitting from them (e.g., Ainsworth, 2006; English & Halford, 1995). There is substantial empirical evidence for this phenomenon: Using multiple representations can foster students' learning, but only if the students are encouraged to actively create connections between these representations (Bodemer & Faust, 2006; Rau et al., 2009; Renkl et al., 2013). Techniques to support learners in integrating multiple representations encompass on the one hand methods that make salient the elements in the different representations which correspond to each other, such as the use of an integrated format or color coding (e.g., Kalyuga et al., 1999; Renkl et al., 2013). On the other hand learning with multiple representations can be fostered by instructional procedures encouraging metacognition using self-explanation prompts or by informing the learners about the function of multiple representations (Renkl et al., 2013). Rau and colleagues (2009) for instance conducted a study with so-called intelligent tutoring systems and found that students learned more with multiple pictorial representations of fractions than with a single pictorial representation – but only when prompted to self-explain how the pictorial representations relate to the symbolic fraction representations. Whereas both kinds of measures can help students to benefit from learning with multiple representations, Renkl et al. (2013) argued that the first kind of techniques can merely support making connections between different representations on a surface level.

The significant role of using multiple representations applies to all kinds of mathematical contents (cf. Kuntze et al., 2011). Nevertheless, it has content-specific facets as well, since different content domains typically use different kinds of representations which have to be integrated by learners (e.g., Acevedo Nistal et al., 2009; Kuhnke, 2013). For the purposes of this study we chose to focus on the content domain of fractions, since it is especially well known that different representations of fractions may highlight different core aspects of the concept and that consequently changing between them is essential (e.g., Ball 1993a; Malle, 2004; cf. also Figure 5.28). In particular making connections between symbolic-numerical representations and appropriate pictorial (diagrams, sketches, illustrations) as well as content-related representations such as real world situations plays a key role for conceptual understanding and sustainable learning of rational numbers (e.g., Hasemann, 1981; Malle, 2004).

The dilemma of teaching with multiple representations

In a general pedagogical sense, teaching has to face several contradicting requirements (Helsper, 1996; Lampert, 1985). For instance classroom situations require individualized reactions which connect to the world of the specific student on the one hand, but the teacher's actions also have to be in line with general principles and goals on the other hand – two potentially conflicting aspects which have to be balanced by the teacher. Such contradictory demands concern not only a general pedagogical level, but they can also be discipline-specific: Ball (1993b) pointed out several dilemmas "which arise out of the contradictions inherent in weaving together respect for mathematics with respect for students" (p. 7). One of the dilemmas she described centers on representing mathematical concepts; she illustrated the difficulties of finding good instructional representations doing justice to both the students' understanding and the mathematical content. A related content-specific dilemma becomes apparent in the light of the above reasoning (cf. Dreher, 2012b): Teachers are responsible for providing their students with a variety of different representations of mathematical concepts as tools for successful mathematical thinking and for encouraging them to integrate these multiple representations. However, the fact that dealing with multiple representations can easily lead to excessive demands for learners, which may result in confusion and frustration instead of conceptual understanding, suggests rather to economize on representations.

In view of its classical meaning the notion "dilemma" as a problem forcing a choice between equally undesirable alternatives may be seen as an exaggeration, since the dissonance between these two conflicting requirements can at least partly and situation-specifically be resolved by using instructional procedures such as those mentioned above to support learners in making connections between multiple representations. In line with this understanding, Lampert (1985) proposed another way of thinking of a dilemma, namely as "an argument between opposing tendencies within oneself in which neither side can come out the winner" (p. 182). She reasoned further that "from this perspective, my job would involve maintaining the tension between my own equally important but conflicting aims without choosing between them" (p. 182). Using this definition of the term, it is possible to balance a dilemma in the sense of not choosing one of its sides in general but acknowledging both of them and dealing with them situation-specifically in a reflexive way.

Hence, our study aims at investigating how teachers manage to balance the dilemma of multiple representations being aid and obstacle for learning mathematics in this sense. The first part, namely acknowledging both sides, may in a first step be looked at on the level of the teachers' views. As this study is set in the domain of fractions, corresponding domain-specific views are focused upon. Typical views that reflect the perception of the side "obstacle for learning" include obviously the concern that too many different representations of fractions could confuse students. Moreover, they encompass the view that having a pictorial "standard representation" is desirable/necessary (cf. e.g., Wagner & Wörn, 2012) and also the worry that highlighting multiple representations of fractions could impede the students' learning of calculation rules for fractions. Acknowledging the side "aid for learning" may, from a discipline-specific perspective, be expressed in the view that using different representations of fractions and changing between them is essential for the students' conceptual understanding of fractions. However, there is at least one further argument for using multiple representations that is often used and which is not discipline-specific, namely the view that students should have the opportunity to choose their preferred representation of fractions in order to account for their individual differences (cf. e.g., Cox, 1999).

Balancing both sides of the dilemma should not mean that little significance is attributed to either side, but instead views reflecting both sides should be held simultaneously, which are weighted against each other and balanced context- and situation-specifically. It depends sensitively on variables such as the particular students, their current thinking, and the specific representations, whether a certain change of representations is rather an aid or an obstacle for students' understanding (cf. e.g., Acevedo Nistal et al., 2009). Consequently, managing to balance the dilemma must go beyond the level of (rather general) views and should become evident in particular in dealing with the dilemma in situated contexts. For this reason, our study does not merely ask whether mathematics teachers are aware of both sides of the dilemma on the level of (domain-specific, but rather global) views, but it focuses moreover on exploring how they acknowledge and balance the two sides context-specifically when they reflect and analyze learning opportunities.

Evaluating the learning potential of tasks

According to results of the TIMSS study, students spent 80 % of their time in the mathematics classroom working on tasks (Hiebert et al., 2003). Moreover, Baumert et al., (2010) found that the learning potential and in particular the potential for cognitive activation of the tasks selected by teachers was a significant predictor of their students' mathematics achievement. Therefore, it is important for teachers to evaluate the learning potential of tasks when they select or create them for the mathematics classroom. The learning potential of a task is certainly a broad construct which accounts for the potential of a task to stimulate learners' cognitive activation in the sense of insightful cognitive learning activities (cf. Baumert et al., 2010; Weideneder & Ufer, 2013). Thus, a task's learning potential may be influenced by many factors, such as the way it challenges students' beliefs or its potential to activate prior knowledge (cf. Baumert et al., 2010).

Since tasks are a practical means of giving students the opportunity to get to know different representations of a mathematical object and of fostering them to make connections between these multiple representations (Duval, 2006), the learning potential of tasks may in particular also be connected to their use of representations. Especially tasks focusing on conversions from one mode of representation to another (and back), which promote insight into their interrelations, have the potential to foster students' understanding (Duval, 2006). Regarding this study, the learning potential of a task from the perspective of dealing with multiple representations is understood in this sense.

The results of a prior study about pre-service teachers' views on pictorial representations in tasks indicate that many pre-service teachers tended to overemphasize the motivational aspect of pictorial representations and hardly saw the learning potential of such pictorial representations which enable students to take an additional approach to mathematical concepts (Kuntze & Dreher, 2014). Similarly, Ball (1993a) reported that American teachers attributed a predominant significance to the motivational potential associated with pictorial representations and appeared to neglect their role for conceptual learning.

Against this background the question arises as to whether mathematics teachers are preoccupied with the idea of using multiple representations in the sense of "adding a potentially motivating picture" or whether they are aware of the learning potential of tasks focusing on conversions of representations. In other words: Do they acknowledge the role of multiple representations as being key for conceptual mathematical understanding which can in particular be beneficial for the learning potential of tasks?

Theme-specific noticing

In order not to let multiple representations become an obstacle for students' understanding in the mathematics classroom a certain sensitivity for the cognitive demands of changing between representations is needed. In particular teachers should realize in which contexts a specific conversion of representations in their instruction and explanations is insightful for the students and can foster their learning and when a conversion is not necessary from the content point of view and rather hindering the learners' understanding. In the light of the more general framework about teacher noticing developed by van Es and Sherin (2002), paying attention to changes of representations in specific instructional situations and evaluating whether they are sensible for students' learning, can be seen as a form of theme-specific noticing (cf. Dreher & Kuntze, 2014). As teachers' noticing became an important subject in the field of mathematics education in the last years, it is in the focus of a growing body of research, for which however somewhat different conceptualizations of noticing are used (Sherin et al., 2011). For this study we use the notion in the way it was specified by van Es and Sherin (2002). Therefore, teachers' noticing is not only about attending to what is significant in the mathematics classroom (selective attention), but it also includes making sense of and evaluating what is observed by drawing on corresponding knowledge and views (knowledge-based reasoning) (cf. Sherin, 2007). From the perspective of the dilemma of using multiple representations in the mathematics classroom it is a key question whether mathematics teachers attend to instances of the student-teacher interaction which are crucial for balancing the dilemma and evaluate them by taking into account situation-specific aspects. Such theme-specific noticing may be characterized as attending to critical conversions of representations in specific classroom situations and evaluating whether they are aid or obstacle for the students' understanding (Dreher & Kuntze, 2014). In particular the question arises whether mathematics teachers are not only aware of the role of multiple representations as being a potential obstacle for understanding on the level of views, but if they also notice when and how such obstacles occur.

Teachers of lower-achieving students facing the double role of multiple representations

It may be assumed that the double role of multiple representations in the mathematics classroom of being aid and obstacle for learning is especially relevant with respect to students who are lower-achieving in mathematics (Kuhnke, 2013). Schipper (2005) described difficulties in conducting conversions of representations as one out of four main characteristics of impairments in arithmetic. Similarly, Moser Opitz (2009) identified such difficulties as a main predictor for low achievement in lower-secondary mathematics. These findings suggest that specifically lower-achieving learners should be encouraged to make connections and changes between representations. However, it is certainly not enough to provide these students with a variety of different representations of a mathematical object as they are often particularly affected by excessive demands induced by multiple representations (e.g. Kuhnke, 2013; Schipper, 2005). Accordingly, an even greater sensitivity regarding the ambiguous role of multiple representations for learning mathematics is needed for teaching lower-achieving students. Hence, one could suppose that mathematics teachers who work with lower-achieving students are more experienced in managing to balance the dilemma of multiple representations being aid and obstacle for learning. In particular it could be expected that these teachers are especially aware of the fact that multiple representations can confuse especially lower-achieving students. Hence, our study compares teachers from two different school types regarding their theme-specific evaluation of tasks and noticing: on the one hand teachers from secondary schools for lower-achieving students ("**H**auptschule"/"**W**erk**r**ealschule") and on the other hand teachers at academic-track secondary schools ("**Gy**mnasium"). In the following, these two groups will be referred to as *HWR teachers* respectively *GY teachers*.

Although the evaluation of the learning potential of tasks and theme-specific noticing may not be the only contexts in which mathematics teachers have to manage balancing the dilemma of multiple representations being aid and obstacle for learning, these two aspects may be seen as being particularly central and are thus the focus of this study.

Research interest

As reported above, there is substantial empirical research highlighting the significant role of using multiple representations for students' learning in the mathematics classroom. Research into aspects of how teachers manage to balance the corresponding dilemma is however scarce. Whereas several studies have assessed teacher professional knowledge about representations concentrating on the selection of representations with respect to their advantages and disadvantages (cf. e.g., Ball et al. 2008, Kunter et al., 2011), there are merely a few studies concerning selected aspects of how teachers deal with the double role of multiple representations for learning mathematics. One example is a qualitative study by Bossé et al. (2011) which focused on teachers' expectations of students being able to perform different conversions of representations. In view of the above reasoning there is hence a need for research regarding the question of how teachers balance the dilemma of multiple representations being aid and obstacle for learning mathematics such as targeted by this study.

In line with this need for research, the evaluations of our study regarding the content domain of fractions are guided by the following research questions:

1. What views about the role of multiple representations for learning mathematics do (German) mathematics teachers have? Do their views indicate awareness of both sides of the dilemma?
2. How do they evaluate the *learning potential* of types of tasks which make use of multiple representations in different ways (conversions of representations vs. unhelpful pictorial representations)?
3. Do the teachers' evaluations of specific classroom situations indicate theme-specific noticing, i.e., do the teachers notice conversions of representations and their *potentially hindering role* for students' understanding?
4. Do GY teachers and HWR teachers differ in how they see the dilemma with respect to their views, their evaluations of the tasks or their theme-specific noticing?
5. Is the teachers' awareness of the two sides of the dilemma on the level of views interrelated with their theme-specific evaluations of tasks and their theme-specific noticing?

Sample and methods

For answering these research questions, a corresponding paper-pencil questionnaire was designed. This questionnaire is based on a previous version, which was tested in a pilot study (Kuntze & Dreher, 2014) and subsequently developed further. At the beginning of the questionnaire explanations of the notions *representation* and *pictorial representation* in a mathematical context were given in order to ensure a similar understanding of all participants regarding these central terms for the study.

The questionnaire was answered by a sample of 102 German in-service teachers, where 77 (35 female, 39 male, 3 without data) were teaching at academic-track secondary schools ("GY teachers") and 25 (15 female, 10 male) were teaching at secondary schools for lower-achieving students ("HWR teachers"). The GY teachers had a mean age of 40.6 years ($SD = 11.8$) and they were teaching mathematics on average for 12.4 years ($SD = 11.5$). The HWR teachers were on average 39.9 years old ($SD = 11.3$) and their mean teaching experience for mathematics was 10.8 years ($SD = 9.5$). The participants completed the questionnaire at their schools in the presence of the first author or a student research assistant and they were given as much time as they needed.

Corresponding to our research questions, three sections of the questionnaire were in the focus of this study: The first section was designed to investigate the participants' views on using multiple representations for teaching fractions. Secondly, the teachers were asked to evaluate the learning potential of six fraction problems by means of multiple-choice items and thirdly a vignette-based format was used to elicit the teachers' theme-specific noticing. In the following, these three sections are described in more detail.

Against the background of the above reasoning about domain-specific views reflecting the perception of the double role of multiple representations for learning fractions, a corresponding questionnaire section was designed which focuses on the five constructs presented in Table 5.9. Regarding each of the items the participants could express their approval or disagreement on a four-point Likert scale. In view of the assumptions that teachers may express one reason for acknowledging a particular side but not the other (e.g., "fear of confusion by multiple representations" but not "multiple representations impede learning rules") and that teachers may in particular hold views reflecting both sides of the dilemma simultaneously, it is expected that these five constructs are empirically separable. The appropriateness of this theory-based model was examined empirically be means of a confirmatory factor analysis (CFA) which is reported in the results section.

The second questionnaire section was designed to provide insight into whether the teachers' perception of the side "aid for learning" of the dilemma becomes evident in the way they evaluate the learning potential of fraction tasks. In particular this section aims to investigate whether the participants recognize the learning potential of tasks focusing on conversions of representations in comparison with tasks including rather unhelpful pictorial representations by asking them to evaluate the learning potential of six fraction tasks. Three of these tasks are about carrying out a conversion of representations, whereas solving the other three tasks means just calculating an addition or a multiplication of fractions on a numerical-symbolical representational level. The pictorial representations that are given in the problems of this second type are not particularly helpful for the solution, since they cannot illustrate the operation needed to carry out the calculation. Samples of both kinds of tasks are shown in Figure 5.29.

Make up a situation or a word problem which is suitable for the calculation $3 \div \frac{1}{4}$ and then use it to solve the calculation.	Do you know what $\frac{1}{2} \times \frac{1}{4}$ is? You can use the pictures below to help: 

Figure 5.29: Samples for tasks of type 1 (left) and of type 2 (right)

Of course these two types of tasks are not representative of all fraction tasks and there are in particular also other kinds of fraction tasks which have a high learning potential. The idea behind this bipolar design is however that contrasting these two types of tasks against each other affords insight into whether and where teachers see the learning potential of multiple representations for fraction tasks. In terms of context information, the teachers

Scale (identifier)	Items
A: Multiple representations (MR) for understanding	It is essential for their understanding that students master using different representations of fractions and that they can change between them. To understand fractions properly, it is necessary to use many different representations in class. Students can only really understand fractions by using many different representations of fractions and by switching between these representations.
B: Multiple representations (MR) for individual preferences	In order to give students the opportunity to choose their preferred type of representation, which they most easily understand, they should be provided with many different representations. Taking the students' individual preferences into account, teachers should present them with as many different pictorial representations for fractions as possible in class. In order to give students the opportunity to choose their preferred type of representation, which they most easily understand, they should be provided with many different representations.
C: One standard representation	It is best to use only one kind of pictorial representation for fractions in lessons, so that you can always come back to this as a 'standard' representation. If I was teaching fractions, I would choose a standard representation for fractions (e.g. pizzas) so that there is a common reference basis for classroom discussions. If I were to teach fractions, I would always use the same type of pictorial representation (e.g. pizzas) so that students could master it and this type of representation could be used again and again.
D: Fear of confusion by multiple representations (MR)	Several different pictorial representations for fractions could confuse students, especially the weaker ones. Weaker students can get confused by having too many different pictorial representations of fractions. The confinement to one type of pictorial representation of fractions is more suitable, especially for weaker students, than a mixture of different representations.
E: Multiple representations (MR) impede learning rules	If students pay too much attention to pictorial representations, their ability to confidently do calculations with fractions is impeded. Too many different pictorial representations of fractions can prevent the students from learning to competently do fraction calculations. If too many pictorial representations are used, the ability to competently do calculations with fractions will be adversely affected.

Table 5.9: Scales regarding views on dealing with multiple representations for teaching fractions

were told that the tasks were designed for an exercise about fractions in school year six. The evaluation of the learning potential of these tasks was carried out by means of the following three multiple-choice items which were used for each of the tasks:

- "The way in which representations are used in this task aids students' understanding."
- "The role that representations play in this problem is such that students' mathematical skills will improve."
- "Students can learn a lot from this problem."

The participants could express their approval or disagreement regarding these items on a four-point Likert scale. The assumption that the teachers' evaluations regarding the learning potential of two tasks of the same type are more similar than those of two tasks of different types suggests that two second order factors can be empirically separated which represent the evaluations of the types of tasks. The analysis of how well this theoretical model fits the data was carried out by a CFA and is reported in the results section.

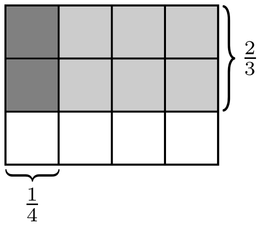
The third questionnaire section was designed to explore whether the teachers' perception of the side "obstacle for learning" of the dilemma becomes evident in their noticing regarding specific classroom situations. For investigating the teachers' theme-specific noticing this study uses a vignette-based design: The participants were given the transcripts of four fictitious classroom situations concerning the topic of fractions. Vignettes in the form of transcripts instead of videos were chosen since they better afford controlling for unintended disturbing factors such as external characteristics of the teacher. All four classroom situations have the following in common: A student makes a comment revealing a misconception or asks a question and thus prompts the teacher to react somehow.

The response by the fictitious teacher uses another representation than the student without making explicit connections between the given and the newly introduced representation. In other words a change of representations is conducted that is potentially hindering for students' understanding and not necessary from the content point of view. Figure 5.30 shows an example of such a fictitious classroom situation. In this case a student asks about how one can see the addition of two fractions in the given rectangle. The teacher however explains the calculation using a pizza representation without making any connections between the two representations. The rectangle representation would in fact be suitable for showing the addition, since the subdivision into twelfths is already given. Therefore, it is neither necessary nor advisable to make the student deal with another representation at this point.

With respect to each of the four classroom situations in this questionnaire section, the teachers were asked the following question: "How much does this response help the student? Please evaluate the use of representations in this situation and give reasons for your answer". It should be noted that by this question the participants were prompted to evaluate the use of representations. In comparison with the possibility to ask them more generally to evaluate the teacher's reaction, this approach has the advantage that the teachers are more likely to show their theme-specific noticing in their answers. A similar methodological approach has also been used in the study by Jacobs et al. (2010) on professional noticing of children's mathematical thinking.

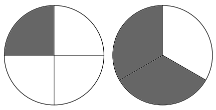
For answering the corresponding research question, the participants' answers were analyzed regarding two main aspects: paying attention to the change of representations and sensitivity to potentially negative effects of this change of representations for the student's understanding. Accordingly, for each answer it was coded in a top-down approach as to whether it shows that the participant has paid attention to the conversion of representations and whether he or she has seen it critically. Those answers for which both is true (i.e., reference to change of representations and negative or balanced evaluation) were considered to provide a rough indicator of the theme-specific noticing that this study targets (cf. e.g., bold answers in Table 5.10). Table 5.10 illustrates this coding by means of sample answers from the data.

T illustrates the calculation $\frac{1}{4} \times \frac{2}{3}$ on the board:

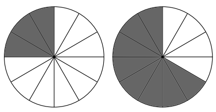


S: And how can you see here what $\frac{1}{4} + \frac{2}{3}$ is?

T: Well, this cannot be seen very well in this picture. For this it would be better to look at pizzas [draws]:



Before we can add the fractions, we have to make all the pieces the same size. Therefore we have to subdivide the pizzas:



Now we see that we have $\frac{3}{12}$ and $\frac{8}{12}$. So, if we add, we get $\frac{3+8}{12} = \frac{11}{12}$.

Figure 5.30: Sample vignette (T: teacher, S: student)

Evaluation	Reference to change of representations	
	No	Yes
None		
Positive	"I find the reaction acceptable and good. Especially for adding fractions, pizzas are still most suitable."	"In this way S realizes that he must proceed differently for adding (he thinks of the pizza) than for multiplying (squares). Choosing a different form of representation is thus reasonable, I think."
Balanced	"For many students this is too difficult, although the addition is well explained in the example."	**"This reaction can help S to answer his question himself, but it will probably rather frustrate him, since there is no response to his question. (. . .) Jumping to a new representation is rather confusing."**
Negative	"The student is confused, since multiplication suddenly turns into addition."	**"T. could and should have shown the addition using the partitioned rectangle. The change hardly helps the student."**

Table 5.10: Coding illustrated by means of sample answers to the vignette shown in Figure 5.30; bold answers were considered to indicate theme-specific noticing

On this basis a corresponding score for theme-specific noticing could be calculated, which counts in how many (out of four) cases the participant's answer indicates that the change of representations and its critical role for the student's understanding was noticed. In the coding procedure all answers were double-coded by the first author and a student research assistant with high inter-rater reliability: Cohen's kappa was 0.87 regarding both dimensions (i.e., "evaluation" and "reference to change of representations"). Discrepancies were resolved through discussion in which an agreement could always be reached.

Results

We start with the results concerning the first research question, namely with the teachers' views about whether multiple representations should be emphasized for teaching fractions. In a first step, a confirmatory factor analysis (CFA) was conducted to establish the underlying structure of the five intended constructs (cf. Table 5.9). The appropriateness of our theory-based model was assessed by several measures of global model fit. The *RMSEA* (root mean square error of approximation) can be interpreted as the amount of information within the empirical covariance matrix that cannot be explained by the proposed model. The model may be classified as acceptable if at most 8 % of the information are not accounted for by the model, i.e. $RMSEA \leq 0.08$ (Kline 2005). The current model meets this criterion as *RMSEA* is 0.064. Furthermore, measures of incremental fit were employed, namely the TLI (Tucker- Lewis index) and the CFI (comparative fit index). An acceptable fit is in both cases indicated by values ≥ 0.90 (Kline 2005). For the proposed model the TLI is 0.93 and the CFI is 0.95 and hence these criteria are met as well. Thus, although the χ^2 statistic is significant ($\chi^2 = 113$, $DF = 80$, $p = 0.01$), it is concluded that the model fits the data reasonably well. As regards the local model fit, the analysis yielded that all factor loadings are highly significant ($p < 0.001$) and the factor reliabilities range from 0.77 to 0.83. In order to examine whether the five constructs in our model are empirically separable, the discriminant validity of each construct with respect to the others was assessed using chi-square difference tests (cf. e.g., Jöreskog, 1971). This means that it was checked for each pair of constructs in our model, if restricting their correlation to 1 makes the fit to the data significantly worse. Since in each case this parameter restriction led to a significantly poorer fit ($p < 0.001$), it may be concluded that all five constructs in our model show sufficient discriminant validity. Hence, for each of the five constructs a corresponding scale could be formed. The means and their standard errors which are presented in Figure 5.31 suggest similar views of HWR teachers and GY teachers: On average, both types of teachers have approved of using multiple representations for fostering their students' understanding and for taking into account individual differences; they were rather not in favor of restricting to one standard representation for fractions and they were not afraid of multiple representations impeding their students' learning calculation rules for fractions. In fact, the comparison of the two subsamples regarding these scales yields no significant differences. In particular, the data do not indicate that the HWR teachers were on average significantly more afraid of confusing their students by using multiple representations than the GY teachers.

As the CFA has shown that there is no single one-dimensional scale on which a teacher's specific views can be located between the perception of the two sides of the dilemma, it is worthwhile to investigate more closely whether there are answering patterns of teachers indicating that those teachers are indeed holding views reflecting both sides simultaneously. For exploring different answering patterns a hierarchical cluster analysis using Ward's method was conducted. Ward's method is based on an analysis of variance where the criterion for fusion of two clusters is to produce the smallest possible increase in the error sum of squares and it is in particular applied when there is no prior knowledge of how many clusters there should be (e.g., Burns & Burns, 2008). The cluster analysis was based on the two scales which express the two sides of the dilemma most explicitly: "multiple representations for understanding" and "fear of confusion by multiple representations".

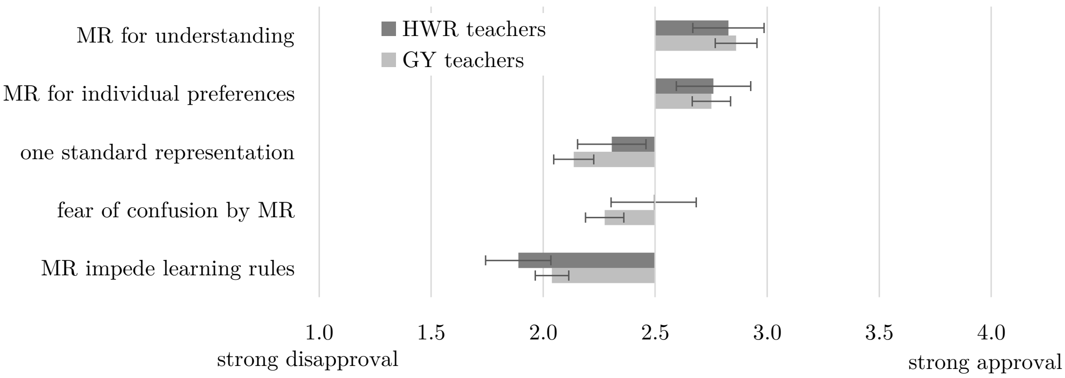


Figure 5.31: Views on the role of MR for teaching fractions (means and their standard errors)

This analysis yielded three clusters showing distinct answering patterns which are presented in Figure 5.32. The results shown in Figure 5.32 indicate that the majority of the participating teachers (47 GY teachers and 14 HWR teachers) emphasized the importance of using multiple representations of fractions for fostering the students' understanding, whereas these teachers did rather not agree with the assertion that multiple representation of fraction could confuse learners. In contrast, the teachers belonging to cluster 2 which amount to about a quarter of the participants (19 GY teachers and 5 HWR teachers) expressed views suggesting that they saw rather the side "obstacle for learning" of the dilemma. The teachers of cluster 3 (11 GY teachers and 6 HWR teachers) approved on average of both statements represented by the two scales, which may be seen as evidence that – on the level of views – this minority of participating teachers was aware of both sides of the dilemma.

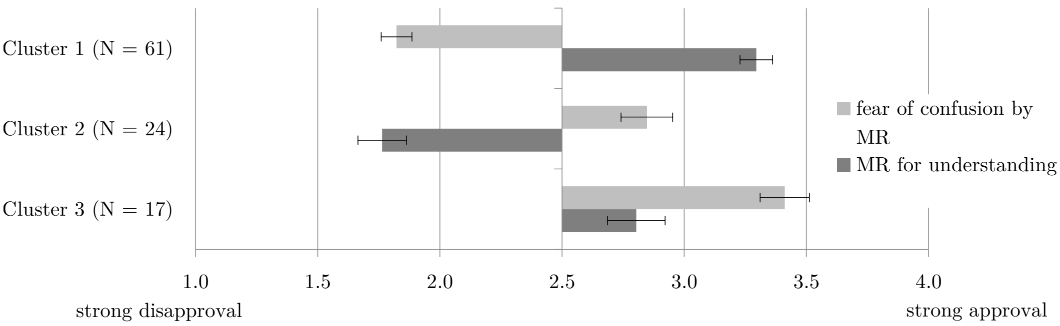


Figure 5.32: Means and their standard errors of the three clusters

Passing on to the question of how teachers acknowledge and balance the two sides of the dilemma in situated contexts when they reflect and analyze learning opportunities we focus next on the second research question, namely the teachers' evaluation of the learning potential of the six tasks given in the questionnaire. Since this evaluation was carried out by means of three items for each of the tasks and these tasks were in turn designed to be examples for two different types of tasks ("conversions of representations" versus "unhelpful pictorial representation") the proposed model for this questionnaire section encompasses $6 \times 3 = 18$ indicators of six first order factors (the evaluations of the tasks) and two second order factors (the evaluations of the types of tasks). Conducting a confirmatory factor analysis (CFA), this model exhibited a reasonably good data fit ($RMSEA = 0.058$, $TLI = 0.92$, $CFI = 0.93$). All the factor loadings are highly significant and both second order factors are reliable with $\alpha = 0.76$ resp. $\alpha = 0.79$. The fact that the chi-square difference test assessing the discriminant validity of the two constructs "evaluation of the learning potential first type of tasks" and "evaluation of the learning potential of the second type of tasks" was highly significant ($p < 0.001$), indicates that the model which distinguishes between the participants' evaluations of the two types of tasks can better predict the data than the corresponding one-dimensional model. Hence, scales regarding the two types of tasks could be formed.

Considering the means and their standard errors presented in Figure 5.33 for each subsample separately yields the following: Firstly, the GY teachers assigned on average a higher learning potential to the first type of tasks ("conversions of representation") ($t(76) = 3.39$, $p = 0.001$, $d = 0.50$) compared to the unhelpful representation tasks; at the same time, they generally evaluated the learning potential of both types of tasks rather positively. Secondly, the HWR teachers showed on average a balanced evaluation of the learning potential of the tasks and there was no significant difference regarding the two types of tasks.

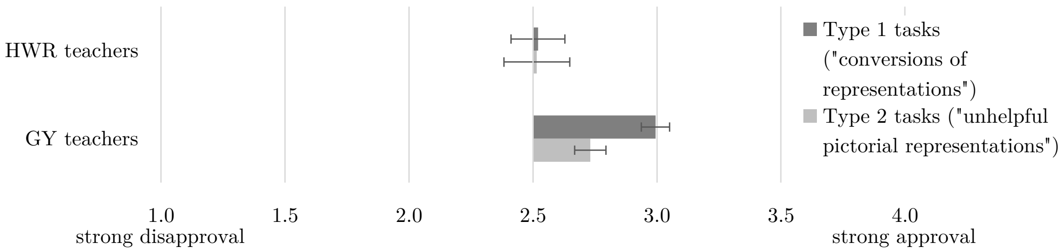


Figure 5.33: Evaluations of the learning potential regarding the two types of tasks

Comparing the evaluations of the two subsamples shows that the GY teachers tended to rate the learning potential of both types of tasks higher than the HWR teachers. However, while the difference regarding the type 2 tasks ("unhelpful pictorial representations") is not significant, the difference with respect to the type 1 tasks ("conversions of representations") is highly significant and represents a large effect size ($t(100) = 4.04$, $p < 0.001$, $d = 0.93$).

Addressing the third research question, we focus next on the teachers' theme-specific noticing: Figure 5.34 shows the mean scores and their standard errors for the two subsamples separately.

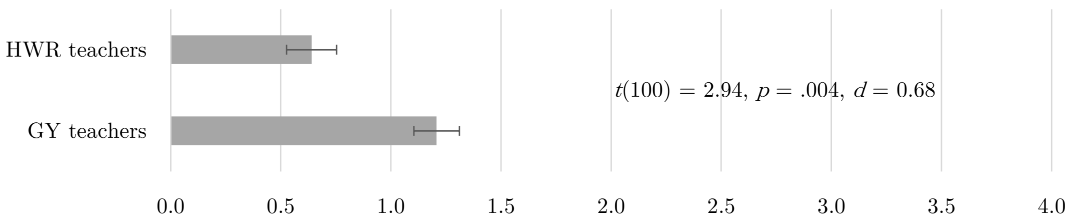


Figure 5.34: Mean number of answers (out of four) that indicate theme-specific noticing

On average, the participants showed a relatively low frequency of noticing conversions of representations and their potentially hindering role for students' understanding. However, the answers of the GY teachers indicated such theme-specific noticing on average almost twice as often as those of the HWR teachers. This significant difference represents a medium effect size (cf. Figure 5.34). Going back one step by considering the participants' evaluations of the single vignettes gives further evidence for this difference of the subsamples with respect to their theme-specific noticing: Regarding each of the four vignettes, in relative terms more HWR teachers than GY teachers evaluated the fictitious teacher's use of representations as being positive. For instance, 32 % of the HWR teachers, but only 9 % of the GY teachers expressed a purely positive evaluation of the teachers' response in the sample vignette presented in Figure 5.30. Moreover (with only one exception, where the percentages were equal) the GY teachers noticed more often the teacher's change of representations. Concerning the sample vignette (cf. Figure 5.30) 79 % of the GY teachers, but only 44 % of the HWR teachers mentioned the change of representations from the "rectangle" to the "pizza".

For finding answers to the fifth research question (which concerns interrelations between the teachers' awareness of the two sides of the dilemma on the level of views and in more situated contexts) it may be insightful to take a further look at the clusters based on the teachers' views. For the corresponding comparison the clusters were restricted to the bigger subsample of GY teachers in order to avoid distorting the result because of the unequal distribution of the two subsamples concerning the three clusters. If being aware of the two sides of the dilemma on the level of views was decisive for evaluating the learning potential of tasks fostering conversions of representations respectively for noticing the potentially hindering role of conversions of representations, one would in particular suppose that cluster 1 ("aid for learning side of the dilemma") and cluster 2 ("obstacle for learning side of the dilemma") differ with respect to these more situated contexts. Considering the means and their standard errors for the evaluation of the type 1 tasks ("conversions of representations") of these clusters however does not indicate any differences. Yet, comparing the clusters with respect to their mean scores regarding theme-specific noticing (cf. Figure 5.35) suggests that the teachers belonging to cluster 1 ("aid for learning side of the dilemma") have on average shown theme-specific noticing more often ($t(49.6) = 1.89$, $p = 0.03$, $d = 0.51$) than their counterparts in cluster 2 ("obstacle for learning side of the dilemma").

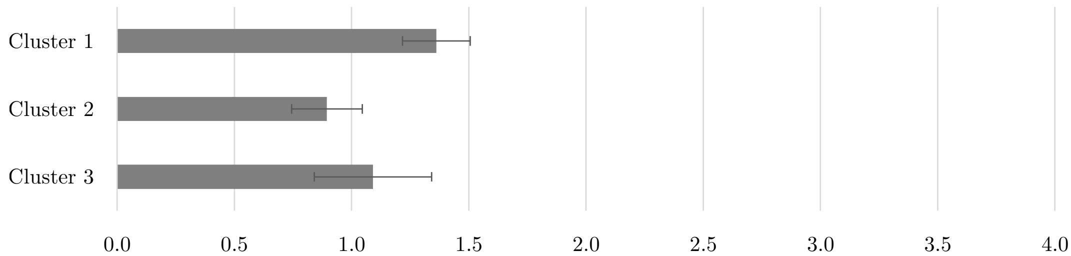


Figure 5.35: Comparison of the three clusters regarding their average score for theme-specific noticing

5.3.1 Discussion and conclusions

The results of this study can give insight into aspects of how teachers deal with the dilemma of multiple representations being aid and obstacle for learning mathematics. This allows in particular to identify prerequisites and specific needs for teacher professional development. The results regarding the first research question suggest that on the level of views about using multiple representations for teaching fractions the majority of the teachers chose one side of the dilemma – most often it was the side "aid for learning". The cluster of teachers indicating views concerning both sides of the dilemma consisted merely of one sixth of the participants. The findings with respect to the fifth research question indicate however that being aware of the dilemma on the level of domain-specific views was not automatically interrelated with acknowledging its two sides context-specifically as captured by the more situated instruments. Such views did not make any significant difference for the theme-specific evaluation of the learning potential of tasks. The differences which were found regarding the theme-specific noticing were rather opposed to the expectation that views marked by high values regarding both sides of the dilemma would correspond with higher theme-specific noticing. The cluster of teachers holding domain-specific views emphasizing the potential of multiple representations for confusion rather than for supporting understanding has noticed "critical" changes of representations on average less frequently than those teachers showing a reverse pattern of such views.

A possible explanation for this phenomenon can be found in the knowledge-in-pieces theory by diSessa (e.g., 1993) which was applied for the conceptualization of teacher knowledge by Kali et al. (2011). Accordingly, people and in particular teachers make sense of events in specific contexts by activating different pieces of knowledge (so-called p-primes) not necessarily maintaining coherence among their beliefs across contexts (Kali et al., 2011).

Furthermore, the findings of this study suggest that looking at teachers' views related to the dilemma of teaching with multiple representations is not sufficient. Thus, even though the structural theoretical approach by Helsper (e.g., 1996) does give hardly any specific recommendations for action, balancing the dilemma must be applied explicitly to the situated contexts of teachers' work. This should in particular be taken into account when designing specific teacher professional development activities.

The need for corresponding teacher professional development is evident from the results regarding the second and the third research questions. For instance: The teachers' theme-specific evaluations of the learning potential of tasks and noticing may be considered non-optimal; it appears that the unsupportive way in which pictorial representations were used in the type 2 tasks remained often undetected as the learning potential of these tasks was on average not seen very critically. Furthermore, the participating teachers showed a relatively low frequency of noticing conversions of representations and their potentially obstructing role for students' understanding.

The findings concerning the fourth research question indicate both similarities as well as clear differences between the two subsamples. With respect to their views on dealing with multiple representations for teaching fractions no significant differences between HWR teachers and GY teachers were found. In particular, there was no difference regarding the view that multiple representations could confuse (especially lower-achieving) students. This is particularly interesting in the light of an earlier comparison between pre-service and in-service teachers where very clear differences regarding the same scales were revealed (Dreher & Kuntze, 2014).

In spite of these very similar views, the results of this study indicate significant differences between the two subsamples in the way they take account of the two sides of the dilemma in more situated contexts. Regarding the teachers' evaluations of the learning potential of types of tasks, the most evident difference is that the GY teachers compared to the HWR teachers attributed on average a clearly higher learning potential to the tasks focusing on conversions of representations. This suggests that the GY teachers' perception of the side "aid for learning" of the dilemma becomes more evident in the way how they evaluate the learning potential of fraction tasks compared to their counterparts teaching at HWR schools. Moreover, the results in terms of the teachers' theme-specific noticing indicate also a higher sensitivity of GY teachers compared to HWR teachers regarding the side "obstacle for learning" in specific classroom situations. To sum up, we may hence conclude that although in particular for teaching lower-achieving students a certain sensitivity for the double role of multiple representations for mathematical understanding may be needed, the HWR teachers showed less such sensitivity than the GY teachers. Further research is needed to explore reasons for these differences between GY teachers and HWR teachers in dealing with the dilemma of multiple representations being aid and obstacle for learning. Such research should in particular also include deepening studies complementing the findings presented here by a qualitative approach which may in particular further affirm the validity of the instruments used in this study. Follow-up research should also be informed by the limitations of this study which suggest interpreting the evidence with care: The study is not representative for German teachers. Furthermore, although the three questionnaire sections were arranged such that the participants completed the part about specific views after the more situated parts, order effects cannot be ruled out. Moreover, even though different aspects of how teachers balance the conflicting requirements associated with using multiple representations were taken into account for the design of the study, its scope is restricted to the content domain of fractions and the constructs can only give an indicator-like insight, but cannot capture a complete picture of the teachers' ways of dealing with the dilemma. In particular the measures used for assessing the context-specific aspects cannot model the full complexity of a mathematics classroom in which teachers usually have to balance the dilemma. Instead these measures were deliberately designed rather plainly to afford first insights into whether teachers acknowledge situation-specifically both sides of the dilemma.

Follow-up research should however also investigate teachers' knowledge and views about dealing with multiple representations in a broader way, addressing in particular also knowledge about possibilities to support learners to integrate multiple representations.

Chapter 6

Discussion & Conclusion

The objective of this dissertation study was to investigate aspects of teachers' views, knowledge, and their noticing about dealing with multiple representations in the mathematics classroom. In particular it was aimed at gaining insights into the interplay of these aspects and into corresponding differences between different groups of mathematics teachers. In this chapter, first, the findings regarding these research interests will be summarized and discussed, then limitations of the study will be considered, and finally the main theoretical and practical contributions and implications will be highlighted, ending with suggestions for further research.

6.1 Discussion of the results

Considering the findings of the three substudies from the perspective of the overall research interest, this section summarizes and discusses the evidence that was found regarding each of the three research interests presented in the third chapter.

First research interest

The first research interest of this dissertation study focused on finding answers to questions as to what views teachers hold about dealing with multiple representations in the mathematics classroom and what they know and notice concerning this matter. In the following, the findings regarding each of the research questions corresponding to this research interest will be discussed.

Regarding the teachers' perception of (global) reasons for using multiple representations in the mathematics classroom, the findings indicate a lack of awareness of the significance that multiple representations have for developing conceptual understanding of mathematics. Both pre-service as well as in-service teachers attached on average significantly less importance to such discipline-specific reasons for using multiple representations compared to other more general reasons. Intriguingly, all the subsamples saw apparently the possibilities to keep students' interest, to take into account their individual preferences, and to support learners to remember concepts as more important purposes of using multiple representations than the key role of conversions of representations for mathematical understanding. This result is not only in line with findings reported by Ball (1993a) and findings from a previous study (Dreher, 2012a), but it can also add to them: This previous study took into account only pre-service teachers and similarly, the data Ball referred to, concerned almost exclusively pre-service teachers. The present study indicates that the phenomenon of overemphasizing non-discipline-specific reasons for using multiple representations, such as motivational purposes, on the level of global views is not restricted to pre-service teachers and only single cases of in-service teachers. This result is particularly problematic insofar as seeing the main purpose of multiple representations in keeping students' interest or in supporting learners' remembering may justify the emphasis of representations that do not foster students' understanding. Also, an overemphasis on taking into account students' individual preferences in learning may prevent teachers from encouraging learners to engage with different representations that are important for the development of an appropriate concept image, since these teachers may want to acknowledge the students' choice of their preferred representation (cf. section 2.5.2).

The findings concerning the teachers' (content domain-specific) views on using multiple representations for teaching fractions may in particular be considered from the perspective of the dilemma of multiple representations being aid and obstacle for learning (cf. section 2.5.3): Whereas some of the scales of the corresponding questionnaire section reflected views emphasizing the side aid for learning (e.g., the view that understanding fractions properly requires using multiple representations), other scales expressed views highlighting the side obstacle for learning (e.g., the view that multiple representations of fractions could confuse students). The results from a cluster analysis in the third substudy suggest that the majority of teachers expressed views that reflect rather one side of the dilemma, whereas only a small minority of the participants appeared to hold simultaneously views referring to both sides of the dilemma. In this context one should bear in mind that it may be possible that participants tried to give a coherent overall picture when they agreed or disagreed with the given items, in the sense of a consistency bias. However, this was apparently not a general problem, since the items of the questionnaire section did not form a single one-dimensional scale and there was a group of teachers that expressed views reflecting both sides of the dilemma. For both subsamples of (German) in-service teachers and also for the English pre-service teachers views reflecting the side aid for learning were on average predominant. The content domain-specific views expressed by both subsamples of German pre-service teachers, however, emphasize on average rather the side obstacle for learning. Despite of the fact that both German in-service teachers as well as English pre-service teachers were on average rather in favor of using multiple representations for teaching fractions, a closer look yields differences: Whereas the English pre-service teachers' approval of using multiple representations of fractions concerned mainly the purpose of taking into account individual preferences of learners, the German in-service teachers put on average comparatively stronger emphasis on the significance of using multiple representations for understanding fractions properly.

The third question regarding this first research interest focused on teachers' task-specific pedagogical content views on using multiple representations of fractions. In particular it was asked as to how mathematics teachers evaluate the learning potential of types of fraction tasks which make use of multiple representations in different ways (conversions of representations vs. unhelpful pictorial representations). The corresponding findings of this study suggest that the unhelpful way in which pictorial representations were used in the second type of tasks remained often undetected, since the learning potential of these tasks was on average not evaluated very critically. The pre-service teachers even tended to assign a higher learning potential to these tasks than to tasks focusing on conversions of representations. In particular, the German in-service teachers at academic track secondary schools were the only subsample that attributed on average a significantly higher learning potential to the type of tasks focusing on conversions of representations than to the calculation tasks with unhelpful pictorial representations. Regarding the question underlying this comparison of the two types of tasks, it may be concluded that many teachers appear to be indeed preoccupied with the idea of using multiple representations in the sense of "adding a potentially motivating picture". In the light of the great significance of mathematics teachers' evaluation of the learning potential of tasks that they use in their classroom (cf. section 2.5.4), this result appears problematic. Tasks are an important means to encourage learners to make connections between multiple representations in order to develop an appropriate concept image (e.g., Duval, 2006). Hence, corresponding task-specific pedagogical content knowledge is needed in order to see the learning potential of tasks that may serve this purpose. There may be different reasons for the fact that many pre-service and in-service teachers attributed a high learning potential to calculation tasks with rather unhelpful pictorial representations: It is for instance possible that these teachers judged by the surface feature of this type of tasks to provide obviously different representation, without examining more closely whether the additional pictorial representations were indeed helpful for solving the tasks. Such a fast judgement could come from overgeneralized views connected with tasks such as "Extra pictorial representations in tasks are always good", but could also stem from a lack of awareness of how significant explicit connections between multiple different representations are for learners' understanding. Another possible reason for a positive evaluation of the learning potential of tasks with unhelpful pictorial representations could be insufficient specific CK: Given the ratings by the participants, it is likely that many teachers thought the pictorial representations could support solving the tasks, since these teachers did not see that the given pictorial representation could not illustrate the calculations the tasks required.

With respect to the participants' theme-specific noticing, the findings of this dissertation study reinforce the impression that mathematics teachers often lack awareness of the crucial role that conversions of representations play for students' conceptual understanding: Overall, the participants' evaluations of fictitious teachers' reactions in specific classroom situations indicated relatively rarely that conversions of representations and their potential to obstruct students' understanding were noticed. By means of a qualitative analysis of cases of teachers' answers that did not show theme-specific noticing, two main reasons could be identified: Either the participants did apparently not pay attention to the conversion of representations or they did realize the conversion, but they did not arrive at a critical evaluation. Examples of the first case suggest that sometimes more dominant views which are connected to the classroom situation may prevent teachers from paying attention to the conversion of representations in the first place. Examples of the second case showed that on the one hand drawing on inappropriate pedagogical content views, but also lacking specific CK could keep teachers from taking a critical stance on the conversions of representations in the classroom situations. Whereas regarding the teachers' evaluations of the learning potential of tasks it could only be speculated about reasons for their on average positive evaluation of tasks with unhelpful pictorial representations, concerning the teachers' theme-specific noticing there is hence evidence indicating that unsuccessful theme-specific noticing is often connected with the teachers' professional knowledge and views.

Specific CK appears to be one of the components of professional knowledge that play a role for teachers' theme-specific noticing and their evaluation of the learning potential of tasks from the perspective of using multiple representation. In particular against this background, the findings of the present study that the participating pre-service teachers and in particular those studying in England showed on average rather poor specific CK, appears highly relevant. Corresponding implications for teacher education will be outlined in the concluding section.

Second research interest

The second research interest of this dissertation study addressed possible interrelations between the aspects of teachers' knowledge, views, and their noticing that were in the focus of the first research interest.

The first of the corresponding research questions regards interrelations between different levels of globality. Investigating such interrelations between pedagogical content views on different levels of globality affords exploring whether such views are consistent: On the one hand global views could be seen as a sort of personal theory applied to more situated contexts, or on the other hand views on different levels of globality could be seen as constructs of their own right, possibly even being in conflict with each other (cf. section 2.4.2). Consequently, the first substudy, which focused mainly on pedagogical content views on using multiple representations, aimed at providing some insight concerning this matter. Accordingly, it was explored whether the global view that multiple representations are essential for conceptual understanding of mathematics is accompanied with corresponding content domain-specific and task-specific views. Considering correlations between the corresponding global and content domain-specific scales yielded significant correlations of however merely moderate effect sizes. Given these effect sizes and in view of cases of participants whose views on these two levels of globality appear to be even contradictory, it may thus not be assumed that global views simply "translate" into content-specific views, but that these views represent constructs of their own right. As regards interrelations of global views and corresponding task-specific views, further inconsistencies were discovered, in particular regarding the subsample of English pre-service teachers: The global view that conversions of representations are important for conceptual understanding of mathematics did not correlate with the evaluation of the tasks focusing on conversions of representations as one could assume, but instead it was significantly positively correlated (moderate effect size) with the evaluation of the learning potential of the tasks with unhelpful pictorial representations. Hence, even though the English pre-service teachers did on average approve of using multiple representations for teaching fractions on the level of content domain-specific views, they could apparently not benefit from such views in terms of evaluating the learning potential of tasks from the perspective of using multiple representations. Instead, such views were connected with positive evaluations of tasks that merely seemed to foster learning with multiple representations on a superficial level. Hence, task-specific pedagogical content knowledge and views facilitating the evaluation of the learning potential of tasks require clearly more than just the conviction that using multiple representations may support students' learning. Evaluating the learning potential of tasks from the perspective of using multiple representations requires in particular an understanding of the conditions under which multiple representations can foster students' learning of mathematics.

Even though no direct correlation between specific CK and task-specific views was found, it is likely that also basic CK about representations in the domain of fractions and their interrelations is relevant for evaluating the learning potential of tasks from the perspective of using multiple representations. As it was pointed out above, in order to implement the goal of fostering students' learning with multiple representations, representations and their connections must be analyzed correctly on the content level.

Whereas the study by Charalambous (2008) could not find any significant correlation between pre-service teachers' noticing of whether connections between multiple representations were made and their CK measured by scales of the MKT test instrument, the findings of the present study can provide some evidence in this regard. Not only did the analysis of cases of pre-service teachers' answers show that insufficient specific CK could prevent them from successful theme-specific noticing, but also the quantitative analysis suggests that the on average relatively low specific CK of the pre-service teachers was a small but significant obstacle to their theme-specific noticing: The pre-service teachers' specific CK was weakly positively correlated with their theme-specific noticing. For the in-service teachers, however, no such correlation was found. In view of the fact that the in-service teachers scored significantly higher in the specific CK test (large effect size), one explanation for this phenomenon may be that the in-service teachers had in overall sufficient specific CK in order to understand the representations and their interrelations necessary for successful theme-specific noticing. Hence, in particular regarding the in-service teachers, not specific CK, but other components of their professional knowledge and views may have played a role for their theme-specific noticing.

Since this study did not only take into account specific CK, but also different components of teachers' PCK, some insight regarding this assumption could be gained. For exploring interrelations between the teachers' theme-specific noticing and aspects of their pedagogical content knowledge and views about using multiple representations, two approaches were taken: On the one hand such interrelations were investigated in terms of analyzing possible correlations between the specific scores or scales. On the other hand, this quantitative approach was complemented by an explorative qualitative analysis of cases of teachers' answers, identifying different components of PCK teachers' drew on for their theme-specific noticing.

Since the theme-specific noticing focused upon in the present study concerns conversions of representations and their evaluation, it is likely that teachers draw on their pedagogical content knowledge and views about the role of conversions of representations for learners' understanding. Consequently, corresponding scales with respect to global and content domain-specific views emphasizing the key role of conversions of representations for students' understanding were considered for the quantitative approach. Accordingly, possible correlations between these two scales on the one hand and theme-specific noticing on the other hand were calculated. Overall, this quantitative approach yielded little interrelations between the teachers' theme-specific noticing and their corresponding global and domain-specific views: Merely the in-service teachers' theme-specific noticing was correlated significantly positively (medium effect) with the global view that conversions of representations are essential for the development of mathematical understanding.

The findings of the complementing qualitative analysis may however offer a possible explanation as to why no strong correlation between the teachers' theme-specific noticing and a certain component of their professional knowledge could be found: The analysis of cases showed that teachers may draw on a variety of different components of their professional knowledge and views for their theme-specific noticing. In particular it was found that such components can be allocated on the whole spectrum of different levels of globality: Situated PCK closely tight to the classroom situation as well as very global professional knowledge regarding conversions of representations not specific to the content domain could result in successful theme-specific noticing. There were cases in which the teachers' answers indicated that their theme-specific noticing was informed mainly by PCK on one level of globality, whereas other teachers' answers showed that they drew on and combined components of PCK on several levels of globality. Hence, these findings indicate that although teachers' noticing is clearly interrelated with their corresponding professional knowledge and views, there is no simple relationship between their theme-specific noticing and a single component of their professional knowledge.

Therefore, these results of the second substudy may contribute to a better understanding of interrelations between teachers' noticing and different components of their professional knowledge. Furthermore, the third substudy of this dissertation project has revisited the issue of interrelations between teachers' pedagogical content views and their theme-specific noticing from the perspective of the dilemma of multiple representations being aid and obstacle for learning mathematics (cf. section 2.5). Since in the first instance, balancing this dilemma may concern the level of views in the sense of generally being aware of both its sides, different profiles of teachers' views on teaching with multiple representations of fractions were explored by means of a cluster analysis. This approach yielded three clusters of teachers, suggesting that most teachers' views rather reflected the side "aid for learning", but that there were also a group of teachers who emphasized the side "obstacle for learning" and a smaller third cluster of teachers who appeared to acknowledge both sides simultaneously. For finding answers to the research question as to whether such awareness of the two sides of the dilemma on the level of views is interrelated with these teachers' theme-specific noticing and their evaluation of tasks, these three clusters were compared to each other regarding their acknowledgement of its two sides in more situated contexts as captured by the corresponding instruments. The results of this analysis show that being aware of the dilemma on the level of domain-specific views in the sense of holding views reflecting both sides was hardly related to these teachers' more situated performance regarding task evaluation and theme-specific noticing. Such views did not make any significant difference for their evaluation of the learning potential of tasks. Moreover, the differences that were found between the three clusters regarding their theme-specific noticing did not support the assumption that holding views reflecting both sides of the dilemma simultaneously would facilitate theme-specific noticing. In fact, the cluster of teachers emphasizing the potential of multiple representations for confusing students noticed "critical" changes of representations on average less frequently than those teachers highlighting the role of multiple representations for fostering learners' understanding.

Bearing in mind the possibility of a consistency bias which may lead participants to weight the importance of different (contradicting) views against each other with the objective of a consistent overall picture, it may be argued that even though the views a teacher expressed reflect merely the side aid for learning, he or she may still have PCK concerning the other side of the dilemma. In this sense, one could draw on the knowledge-in-pieces theory (diSessa, 1993; Kali et al., 2011) that was referred to in section 2.4.2 and argue that teachers evaluate instances in specific classroom situations by activating different pieces of their professional knowledge which can also represent conflicting views. Teachers may thus have drawn on their PCK about potential obstacles for students' understanding connected with conversions of representations, even though they did not express corresponding pedagogical content views. The finding that teachers emphasizing the significant role of multiple representations for the development of students' conceptual understanding for mathematics showed successful theme-specific noticing on average more often than those worrying predominantly about the negative effects of using multiple representations could however also give rise to another possible explanation: Maybe the general awareness of the significance of using multiple representations for fostering students' mathematics understanding serves dealing responsibly with multiple representations in the mathematical classroom better than a rather rejectionist stance regarding multiple representations for the sake of not confusing learners. This would however require that these teachers focusing generally rather on the potential of multiple representations for fostering students' understanding are nevertheless able to recognize situation-specifically problematic aspects of using multiple representations and to support learners in overcoming such obstacles.

Third research interest

The third research interest of this dissertation study was motivated in section 2.6 and focused on comparing different subsamples in order to explore whether the constructs vary across these different groups of teachers.

As it was outlined in section 2.6.1, there is good reason to assume that pre-service teachers' views on dealing with multiple representations in the mathematics classroom are influenced by characteristics of mathematics teaching in their countries. Taking a closer look at such culture-dependent aspects of pre-service teachers' views affords designing learning opportunities for their teacher education which may be more valid within the scope of their cultural setting. Hence, English and German pre-service teachers were compared with respect to their views on using multiple representation in the mathematics classroom.

Regarding global views on purposes of using multiple representations in the mathematics classroom, the only significant difference between the two subsamples lies in the finding that the German pre-service teachers attached significantly more importance to the possibility of supporting learners to remember facts better. Considering content domain-specific views on how to use multiple representations for teaching fractions, however, more differences between the English and the German pre-service teachers became evident. In particular, the English pre-service teachers put significantly greater emphasis on using multiple representations of fractions in order to take into account individual differences of learners. The German pre-service teachers on the other hand rather feared that multiple representations of fractions could confuse learners and impede their learning of calculation rules for fractions. Moreover, they were on average more in favor of using one standard representation than their English counterparts. Interestingly, these differences in the pre-service teachers' views on dealing with multiple representations in the mathematics classroom reflect to a certain extent indeed characteristics and ideas of mathematics teaching in their countries as they were described by Kaiser (2002) and outlined in section 2.6.1. This concerns in particular the high priority of the individual in English classrooms and the greater emphasis on rules and common notations in German classrooms. Furthermore, these results are also in line with findings by Pepin (1999) regarding more general views of teachers in England and Germany and they may in particular add specific aspects to these findings. Hence, these results about differences in the specific views of English and German pre-service teachers may also be seen as positive feedback regarding the culture-sensitivity and validity of the questionnaire instruments that were developed for assessing teachers' views on dealing with multiple representations in the mathematics classroom.

However, the English and German pre-service teachers may not only be compared regarding their global and content domain-specific views on using multiple representations, but also with respect to interrelations of such views with their evaluations of the learning potential of fraction tasks. As it was described in the discussion of findings about interrelations of views on different levels of globality, correlations of such task-specific views with global views on reasons for using multiple representations indicate inconsistencies in particular regarding the English pre-service teachers: Greater emphasis of the significance of multiple representations for building up conceptual understanding of mathematics was significantly related to a more positive evaluation of the learning potential of the calculation tasks with unhelpful pictorial representations. In view of the finding that the English participants had on average very low scores in the specific CK test, one may assume that many of them did not detect that these pictorial representations could not illustrate the operations that were needed to solve the tasks. Instead, they may only have noticed that pictorial representations were additionally given, which may be used by learners optionally according to their individual preferences. Consequently, these English pre-service teachers may need a strengthened CK background in order to give them the opportunity to transfer their positive views about using multiple representations into appropriate evaluations of the learning potential of tasks from the perspective of using multiple representations. The German participants, however, may need in particular learning opportunities fostering their awareness of the discipline-specific reasons for using multiple representations and of their essential role for learners' understanding.

The second substudy compared in-service and pre-service teachers with respect to their knowledge, views, and noticing about using multiple representations in the mathematics classroom. Regarding specific global views the findings suggest that both pre-service teachers as well as in-service teachers put on average less emphasis on the special role of multiple representations for fostering the students' understanding of mathematical concepts compared to reasons that are rather not discipline-specific. Considering more situated views on dealing with multiple representations of fractions, however, yielded clear differences between the participating pre-service and in-service teachers: The in-service teachers were on average less afraid of multiple representations confusing their students or impeding their students' learning of calculation rules for fractions and they were significantly less inclined to using a single pictorial standard representations for fractions than the pre-service teachers. The difference between the two subsamples which represents the biggest effect size ($d = 0.85$) and which is most relevant from the perspective of the theoretical background of this study concerns the scale "multiple representations of fractions for understanding": The in-service teachers attached on average significantly greater importance to the role of conversions of representations of fractions for the students' mathematical understanding than the pre-service teachers. Hence, even though the teachers showed rather little awareness of such discipline-specific reasons on a global level, these findings suggest that experiences in teaching fractions may have facilitated corresponding domain-specific pedagogical content knowledge.

Focusing on the teachers' theme-specific noticing revealed further differences between the pre-service teachers and the in-service teachers who took part in the study: The in-service teachers' evaluations of specific classroom situations indicated theme-specific noticing on average about twice as often as those by pre-service teachers. In view of the fact that the ability to notice is often seen as a distinguishing feature of expert teachers in comparison with novices, this finding may be seen as a piece of quantitative evidence and hence it can add to corresponding results by previous research studies specifically from the perspective of dealing with multiple representations (e.g., Ainley & Luntley, 2007; Berliner, 1994; Jacobs et al., 2010).

Furthermore, the findings of this study suggest that there are differences between pre-service and in-service teachers in the way their theme-specific noticing is interrelated with their corresponding professional knowledge: Merely with respect to the in-service teachers, theme-specific noticing was significantly related to the global view that the ability to perform conversions of representations is key for building up conceptual understanding of mathematics. This may indicate a comparatively stronger connection between the in-service teachers' theme-specific noticing and their corresponding PCK and moreover it underpins the assumption that for the development of teachers' expertise a growth of interrelations between different components of their professional knowledge may play an important role.

However, not all in-service mathematics teachers may be experts in dealing with multiple representations. As it was argued in section 2.6.3, it is in particular likely that it makes a difference whether a teacher works with lower- or higher-achieving students in mathematics. Consequently, the third substudy explored whether teachers at academic track secondary schools (GY) differ from their colleagues at secondary schools for lower-achieving students (HWR) with respect to their specific views, their evaluations of tasks or their theme-specific noticing. The findings of this study indicate both similarities as well as significant differences between the GY teachers and the HWR teachers. On the level of views on dealing with multiple representations of fractions no significant differences between the two subsamples were found. The views that were expressed suggest that both GY teachers and HWR teachers tended on average to rather choose the side aid for learning of the dilemma. In particular, HWR teachers and GY teachers showed no significant differences in their views regarding the potential of multiple representations to confuse students: On average neither the GY teachers nor the HWR teachers emphasized this view. This result is specifically interesting in view of the fact that concerning such content domain-specific views clear differences became apparent regarding the comparisons between English and German pre-service teachers as well as between (German) pre-service and in-service teachers.

Despite of these similar profiles of content domain-specific views, the findings suggest significant differences between the GY teachers and the HWR teachers regarding their evaluations of the learning potential of tasks and with respect to their theme-specific noticing. Considering their task-specific views, the biggest difference between the subsamples concerns their evaluations of the tasks focusing on conversions of representations: On average, the GY teachers evaluated the learning potential of this type of tasks significantly more positively (large effect size) than the HWR teachers. Hence, these teachers were apparently more aware of the potential of these tasks to foster students' mathematical understanding. The finding that the GY teachers' evaluations of specific classroom situations indicated significantly more frequently theme-specific noticing than those by HWR teachers, suggests moreover that the GY teachers showed also comparatively more sensitivity regarding the potentially obstructing role of conversions of representations for students' understanding. In relative terms, the GY teachers paid more often attention to conversions of representations in the specific classroom situations and they gave less often a purely positive evaluation of the fictitious teachers' response than the HWR teachers. It may hence be concluded that the results of the present study do not support the assumption that HWR teachers show more sensitivity for the conflicting requirements arising from the double role of multiple representations for learning mathematics (cf. section 2.6.3). Instead, the GY teachers' performance in the situated contexts as assessed by the instruments of this study indicates comparatively more such sensitivity regarding both sides of the dilemma. This result may also be seen as a further piece of evidence for the need to distinguish between teachers' professional knowledge on different levels of globality, not only with respect to pre-service teachers, but also regarding in-service teachers: Even though on the level of domain-specific views both GY teachers as well as HWR teachers appeared to acknowledge the key role of multiple representations for mathematical understanding, their more situated views regarding specific tasks and classroom situations were obviously different.

6.2 Limitations of the study

Before the main theoretical and practical implications of this dissertation study will be highlighted in the last section, I would like to emphasize the limitations of this study that suggest to interpret the evidence carefully.

First of all, the study is not representative for the populations of teachers that were studied and therefore the possibility to make broader generalizations from the results is limited.

Since the domain of fractions was selected for the domain-specific parts of the study, the scope of the findings is mainly restricted to this content domain. Furthermore, even though a spectrum of different facets of teachers' views, knowledge, and their noticing about dealing with multiple representations in the mathematics classroom was taken into account for the design of the study, the selection of these constructs can merely give an indicator-like insight. In particular, the complementing qualitative analysis regarding cases of teachers' noticing indicated that the participants drew on a broad variety of components of their professional knowledge and views for their theme-specific noticing. Thus, more aspects of teachers' professional knowledge and views may play a role than the quantitative part of this study could assess as constructs.

Further limitations of the study arise from using a paper-pencil questionnaire instrument to explore aspects of teachers' views, knowledge, and their noticing. In particular the instruments used to elicit and capture context-specific aspects such as theme-specific noticing cannot model the full complexity of a real mathematics classroom in which teachers ultimately have to show such noticing. Instead, these measures were deliberately designed theory-based and with reduced complexity. This approach afforded studying these aspects not only in a qualitative, but in particular also in a quantitative way and to gain first insights regarding aspects and interrelations of the mathematics teachers' views, knowledge, and noticing from the perspective of dealing with multiple representations. Also connected with the use of the questionnaire instrument may be possible order effects: Even though the questionnaire sections were arranged such that the participants completed the parts about their content domain-specific and global views after the more situated parts about evaluations of tasks and theme-specific noticing, order effects cannot be ruled out.

Bearing in mind these limitations, the findings of this dissertation study indicate however several aspects of theoretical and practical relevance. Moreover, these constraints of the study give rise to suggestions for future studies which may overcome some of its limitations. Such implications as well as need for further research will be addressed in the following final section.

6.3 Implications of the findings and need for further research

Corresponding to the three research interests, the findings of this dissertation study may serve three main purposes: Firstly, they can provide insight into aspects of mathematics teachers' knowledge, views, and their theme-specific noticing regarding the role of multiple representations for learning mathematics in the sense of identifying specific prerequisites and needs for teacher education and professional development. Secondly, they can contribute to a better understanding of interrelations between different aspects of teachers' professional knowledge, views, and their noticing. And thirdly, the comparison of different groups of teachers allows to identify differences connected with teaching experience, cultural background, and teaching at different school types and to review corresponding assumptions within the scope of the study empirically.

Concerning prerequisites of (German) pre-service teachers, the findings about their global as well as their domain-specific views on using multiple representations in the mathematics classroom indicate lacking awareness regarding the essential role of multiple representations for learning mathematics. Their on average rather positive evaluation of tasks with unhelpful pictorial representations suggests that they did not arrive at the conclusion that the fact that the given representations do not match should be seen critically. As it was discussed above, reasons for this may be an overemphasis on motivational aspects connected with the use of multiple representations or insufficient PCK about conditions under which learners can benefit from using multiple representations. In view of the pre-service teachers relatively poor specific CK it is also likely that they often did not realize that the given pictorial representations could not support solving the tasks. For some of these pre-service teachers a lack of specific CK may also have been an obstacle to theme-specific noticing: The participating pre-service teachers noticed on average relatively rarely conversions of representations and their potentially hindering role for students understanding in specific classroom situations.

Hence, the findings of the present study clearly point to a need for specific learning opportunities for these pre-service teachers in the course of their teacher education. Such learning opportunities should not only focus on professional knowledge on a global level, but they should also integrate situated contexts of dealing with multiple representations, for instance by working on video-taped classroom situations and instructional material that focus on specific contents (see Dreher & Kuntze, 2012). In conjunction with such instructional material and classroom situations, also corresponding CK about multiple representations in the respective content domains should be fostered. Moreover, an important aspect of these situated learning approaches should be making connections to theory about the role of multiple representations for learning mathematics. Accordingly, such learning environments can offer possibilities for pre-service teachers to connect theory-based planning and reflection to situated contexts and thus may foster building up coherent professional knowledge about dealing with multiple representations in the mathematics classroom across different levels of globality. Corresponding professional development courses for teacher education were designed and have already been implemented at Ludwigsburg University of Education (Dreher & Kuntze, 2012).

Findings concerning the in-service teachers investigated by the present study suggest that these teachers differ from the pre-service teachers particularly with respect to the more situated aspects of dealing with multiple representations, especially regarding their theme-specific noticing. Nevertheless, the results indicate as well that experienced teachers are not necessarily experts in using multiple representations in the mathematics classroom. In particular on a global level of views they appear to lack awareness for the discipline-specific role that multiple representations play for conceptual understanding of mathematics, in the sense of an overarching idea that facilitates theory-driven planning and reflecting of mathematics classrooms (see Kuntze et al., 2011). Consequently, implications of these results encompass in particular the need for specific professional development opportunities focusing on this overarching idea for teaching mathematics. Such professional development could aim at connecting the teachers' own experience in their mathematics classrooms in the sense of situated professional knowledge with a theoretical perspective on how to support students in learning with multiple representations of mathematical objects. In-service teachers could benefit from such a theoretical perspective by embedding their situated knowledge into a framework of more global professional knowledge which can then serve in turn as a lens for their theme-specific noticing and for designing rich learning opportunities for their students.

Considering interrelations between components of teachers' professional knowledge and views on different levels of globality, the results of this dissertation study suggest that there is no simple translation between global and more situated professional knowledge. In particular with respect to pre-service teachers, corresponding aspects of their knowledge and views may apparently even contradict each other. Consequently, the findings of this study underpin the assumption that such components of teachers' professional knowledge on different levels of globality may be seen as constructs of their own right (Kuntze, 2012). From a practical point of view, this suggests that it is not sufficient for teacher professional development to address PCK on a global level, but instead it is highly relevant to make consequences regarding specific contents and classroom situations a subject of discussion. This means that within situated contexts such as video-taped classrooms or instructional material it should be discussed whether specific conversions of representations are rather aid or obstacle for students' learning in order to support teachers in balancing the corresponding dilemma situation-specifically in a reflected way.

Regarding the interplay of theme-specific noticing and corresponding professional knowledge, the results of the present study can contribute to a better understanding insofar as it was shown that teachers' may draw on different components of their professional knowledge for their noticing. In particular such components could be located on different levels of globality. Hence, although theme-specific noticing is apparently informed by the teachers' professional knowledge, it is difficult to capture such interrelations in the sense of correlations between theme-specific noticing and specific components of professional knowledge.

The comparison of English and German pre-service teachers with respect to their views on using multiple representations in the mathematics classroom yielded common needs for their teacher education, but it could also underpin the need to take into account culture-specific aspects of such views, which should be beard in mind regarding the design of specific learning opportunities.

Further specific needs for professional development were indicated by the results of the comparison of GY teachers and HWR teachers. Although teaching lower-attaining students may require even more sensitivity regarding the double role of multiple representations for learning mathematics, the participating HWR teachers who are teaching such students showed on average less such sensitivity than the GY teachers. Consequently, the need for specific professional development as pointed out above may apply in particular to HWR teachers.

The findings of this dissertation study may serve as a starting point for future research projects in order to explore particular aspects of teachers' knowledge, views, and noticing regarding dealing with multiple representations in greater depth or aim at a broader perspective by including more facets or content domains.

Investigating particular aspects in greater depth could for instance mean to focus on the teachers' theme-specific noticing and to elicit such noticing not only by text-vignettes, but also by video-vignettes, which may represent classroom situations in a more realistic way. Such a design would also afford insights into what difference it makes whether one or the other methodological approach is chosen for a study. Taking a closer look at certain aspects could also mean to conduct further complementary qualitative in-depth studies. It could for instance be insightful to assess teachers' task-specific views additionally by means of interviews, where reasons for their evaluations of the learning potential of tasks regarding their use of multiple representations could become more apparent.

Further research should also explore how aspects of teachers' knowledge, views, and their theme-specific noticing targeted by this study develop in the course of teacher professional development and with growing teaching experience by means of a longitudinal research design. Moreover, in this context it should also be investigated to which extent such professional knowledge and noticing regarding dealing with multiple representations can be enhanced through specific professional development courses.

And finally, a further highly relevant research interest concerns the question as to what impact aspects of teachers' professional knowledge and their theme-specific noticing regarding the role of multiple representations in the mathematics classrooms may have on their students' competencies in dealing with multiple representations. Consequently, finding answers to this question should be an aim for future research studies.

Appendix A

Überblick über Artikel und Konferenzbeiträge

Im Rahmen dieser Dissertationsarbeit entstanden die folgenden Artikel und Konferenzbeiträge:

Dreher, A., Kuntze, S., & Lerman, S. (2015). Why use multiple representation in the mathematics classroom? Views of English and German pre-service teachers. *International Journal of Science and Mathematics Education*. Advance online publication. doi: 10.1007/s10763-015-9633-6.

Dreher, A., & Kuntze, S. (2015). Teachers' professional knowledge and noticing: The case of multiple representations in the mathematics classroom. *Educational Studies in Mathematics, 88*(1), 89–114.

Dreher, A., & Kuntze, S. (2014). Teachers facing the dilemma of multiple representations being aid and obstacle for learning: Evaluations of tasks and theme-specific noticing. *Journal für Mathematik-Didaktik*. Advance online publication. doi: 10.1007/s13138-014-0068-3.

Kuntze, S., & Dreher, A. (2014). PCK and the awareness of affective aspects reflected in teachers' views about learning opportunities – A conflict? In B. Pepin, & B. Rösken-Winter (Hrsg.). *From beliefs to dynamic affect systems in mathematics education: Exploring a mosaic of relationships and interactions* (S. 295–318). Advance online publication. doi: 10.1007/978-3-319-06808-4_15.

Dreher, A., Nowinska, E., & Kuntze, S. (2013). Awareness of dealing with multiple representations in the mathematics classroom – A study with teachers in Poland and Germany. In: A. Lindmeier, A. Heinze (Hrsg.), *Proceedings of the 37th Conference of the International Group for the Psychology of Mathematics Education* (Vol. 2, S. 249–256). Kiel, Germany: PME.

Dreher, A., & Kuntze, S. (2013). Pre-service and in-service teachers' views on the learning potential of tasks – Does specific content knowledge matter? In: B. Ubuz, Hasar, C; & M. A. Mariotti (Hrsg.), *Proceedings of CERME 8, Antalya* (S. 3035–3044). Ankara, Turkey: Middle East Technical University.

Dreher, A., Kuntze, S., & Lerman, S. (2012). Pre-service teachers' views on using multiple representations in mathematics classrooms – An inter-cultural study. In Tso,T. Y. (Hrsg.), *Proceedings of the 36th Conference of the International Group for the Psychology of Mathematics Education* (Vol. 2, S. 211–218). Taipei, Taiwan: PME.

Dreher, A., & Kuntze, S. (2012a). Pre-service teachers' views about pictorial representations in tasks. In S.J. Cho (Ed.), *The proceedings of the 12th International Congress on Mathematical Education* (S. 5705–5713). Seoul, Korea: ICME.

Dreher, A., & Kuntze, S. (2012b). *The challenge of situatedness in pre-service teacher education – Adapting elements of lesson study in the context of a course on "using multiple representations".* doi: 10.13140/2.1.2341.0881

Dreher, A. (2012a). Vorstellungen von Lehramtsstudierenden zum Nutzen vielfältiger Darstellungen im Mathematikunterricht. *Beiträge zum Mathematikunterricht 2012.* Münster: WTM-Verlag.

Dreher, A. (2012b). Den Wechsel von Darstellungsformen fördern und fordern oder eher vermeiden? – Über ein Dilemma im Mathematikunterricht. In Sprenger, J.; Wagner, A. & Zimmermann, M. (Hrsg.), *Mathematik lernen, darstellen, deuten verstehen – Didaktische Sichtweisen vom Kindergarten bis zur Hochschule.* (S. 215–225) Wiesbaden: Springer Spektrum.

Dreher, A., Till, C., Käfer, O., Schmeißer, S., & Kuntze, S. (2012). Eine Idee – unterschiedliche Inhalte: „Multi-Pack"-Lernumgebungen. *mathematik lehren, 173.*

Dreher, A., & Winkler, J. (2011). Lernumgebung zur Big Idea „Vielfältige Darstellungen nutzen". In S. Kuntze & A. Dreher (Hrsg.), *Big Ideas im Zentrum des Mathematikunterrichts – Fachdidaktischer Hintergrund, Anregungen für die Unterrichtspraxis und Materialien für schüler(innen)zentrierte Lernumgebungen* (S. 23–34). Ludwigsburg: Päd. Hochschule.

Appendix B

Deutsche Zusammenfassung

Die Rolle vielfältiger Repräsentationen für das Lernen von Mathematik wurde in den letzten Jahrzehnten von zahlreichen empirischen Studien in den Blick genommen. Infolgedessen besteht unter Wissenschaftlerinnen und Wissenschaftlern im Bereich der Mathematikdidaktik weitgehend Konsens darüber, dass das Nutzen vielfältiger Repräsentationen eine Schlüsselrolle beim Lehren und Lernen von Mathematik spielt. Während sich eine Reihe empirischer Forschungsprojekte mit dem Lernen von Schülerinnen und Schülern mit vielfältigen Repräsentationen beschäftigt hat, gibt es bislang kaum Studien, die die Rolle der Lehrkräfte in diesem Zusammenhang in den Blick genommen haben. Dementsprechend befasst sich die vorliegende Dissertationsstudie mit der Frage, inwiefern angehende und praktizierende Lehrpersonen sich dieser Schlüsselrolle von vielfältigen Repräsentationen im Mathematikunterricht bewusst sind und welches Wissen beziehungsweise welche Sichtweisen sie diesbezüglich haben. In diesem Zusammenhang wurden nicht nur verschiedene Komponenten professionellen Wissens und entsprechender Sichtweisen untersucht, sondern auch spezifisches Noticing, nämlich, ob Lehrkräfte inhaltlich nicht notwendige und für das Verständnis von Lernenden potentiell hinderliche Repräsentationswechsel in konkreten Unterrichtssituationen bemerken und kritisch bewerten. Auf der Grundlage eines mehrdimensionalen Modells für professionelles Wissen von Lehrkräften befasst sich diese Studie insbesondere damit, wie solche verschiedenen Aspekte professionellen Wissens zusammenhängen und inwiefern sie eine Rolle für das spezifische Noticing von Lehrkräften spielen. Angesichts der großen Bedeutung, die vielfältige Repräsentationen insbesondere im Bereich Brüche haben, wurde für die inhaltsbereichsspezifischen Erhebungsteile der Studie dieser Inhaltsbereich gewählt.

Die Forschungsinteressen der vorliegenden Arbeit wurden im Rahmen von drei Teilstudien umgesetzt, wobei jeweils zwei verschiedene Teilstichproben untersucht wurden: englische und deutsche angehende Lehrkräfte, angehende und praktizierende Lehrkräfte, bzw. Lehrkräfte an Gymnasien und an Haupt-/Werkrealschulen. Dieses Design ermöglicht es, die potentiellen Einflussfaktoren kultureller Hintergrund in verschiedenen Ländern, unterschiedliche Professionalisierungsstadien und Schulkultur in einer explorativen Herangehensweise zu berücksichtigen.

Die unterschiedlichen Aspekte spezifischen professionellen Wissens und entsprechender Sichtweisen von Lehrpersonen wurden mit Hilfe eines Fragebogeninstruments erhoben. Für die Erhebung des spezifischen Noticings der Lehrkräfte wurde ein vignettenbasiertes Design verwendet. Zur Analyse der Daten wurden hauptsächlich quantitative Methoden verwendet. Diese Auswertung wurde jedoch durch eine vertiefende qualitative Analyse von Fällen ergänzt, um explorativ zu untersuchen, auf welche Komponenten professionellen Wissens bzw. auf welche Sichtweisen die Lehrkräfte bei ihrem Noticing zurückgriffen.

Die Ergebnisse dieser Studie legen nahe, dass die Teilnehmerinnen und Teilnehmer die Schlüsselrolle vielfältiger Repräsentationen für das Lernen von Mathematik im Sinne ihrer disziplinspezifischen Bedeutung nicht in vollem Maße erkannt haben. Dies weist auf einen Bedarf an Angeboten spezieller Lernumgebungen im Rahmen der Aus- und Fortbildung von Lehrkräften hin. Unterschiede zwischen den Teilstichproben wurden besonders im Hinblick auf eher situiertes professionelles Wissen und Noticing zum Umgang mit vielfältigen Repräsentationen deutlich. Des Weiteren stützen die Ergebnisse die Annahme, dass bezüglich des Spektrums zwischen situiertem und globalem Professionswissen von Lehrkräften verschiedene Komponenten unterschieden werden können. Die Resultate weisen insbesondere darauf hin, dass all diese Komponenten eine Rolle für das spezifische Noticing der Lehrpersonen spielen können.

Im Folgenden wird überblicksweise der theoretische Hintergrund der Studie dargestellt, woraus dann das Forschungsinteresse der Dissertationsstudie abgeleitet wird. Anschließend wird zusammenfassend das Design der Studie beschrieben, bevor schließlich die wichtigsten Ergebnisse und Folgerungen berichtet und diskutiert werden.

Theoretischer Hintergrund

Die Rolle vielfältiger Repräsentationen für das Lernen von Mathematik

Sowohl Lernende als auch Experten sind auf die Verwendung von Repräsentationen angewiesen, wenn sie Mathematik betreiben, denn mathematische Objekte sind ohne diese nicht zugänglich (Duval, 2006; Janvier, 1987; Mason, 1987). Eine Repräsentation wird dabei verstanden als etwas, das für etwas anderes steht (vgl. Duval, 2006; Goldin & Shteingold, 2001). Repräsentationen können jedoch meist jeweils nur gewisse Aspekte und Eigenschaften des mathematischen Objekts, für das sie stehen, sichtbar machen. Folglich werden in der Regel mehrere solcher Repräsentationen benötigt, die sich gegenseitig ergänzen, um ein flexibel nutzbares mathematisches Begriffsverständnis aufzubauen (Ainsworth, 1998; Duval, 2006; Even, 1990; Goldin & Shteingold, 2001; Janvier, 1987; Tall, 1988). So ist es beispielsweise im Inhaltsbereich Brüche nicht ausreichend, auf bildliche Repräsentationen in Form von Kreisdiagrammen zurückzugreifen, da diese zwar den Bruchzahlaspekt „Teil eines Ganzen“ deutlich machen, jedoch zum Beispiel nicht den Operatoraspekt, der für die Multiplikation von Brüchen benötigt wird (Wittmann, 2006). Folglich spielen das Nutzen vielfältiger Repräsentationen und Repräsentationswechsel eine wichtige Rolle für erfolgreiches mathematisches Denken und Problemlösen (Ainsworth, Bibby & Wood, 2002; Duval, 2006; Elia, Panaoura, Eracleous & Gagatsis, 2007; Even, 1998). Dementsprechend wird die Schlüsselrolle vielfältiger Repräsentationen für das Lernen von Mathematik auch in den Bildungsstandards für Mathematik in vielen Ländern hervorgehoben (vgl. z.B. KMK, 2003; NCTM, 2000). So beschreiben die KMK-Bildungsstandards „mathematische Darstellungen verwenden“ als eine von sechs allgemeinen mathematischen Kompetenzen, die insbesondere das Erkennen von Beziehungen und das Wechseln zwischen unterschiedlichen Repräsentationsformen umfasst (KMK, 2003, S. 8).

Der Umgang mit vielfältigen Repräsentationen im Mathematikunterricht kann Lernenden jedoch auch Schwierigkeiten bereiten: Eine Reihe von empirischen Untersuchungen hat gezeigt, dass dies soweit führen kann, dass vielfältige Repräsentationen das Lernen eher behindern als unterstützen (Ainsworth et al., 2002; Tabachneck, Leonardo & Simon, 1994). Da eine Repräsentation nicht offensichtlich und selbsterklärend für ein mathematisches Objekt steht, sondern diese Verbindung Interpretations- und Aushandlungsprozessen unterliegt (Cobb, 2002; Gravemeijer, Lehrer, van Oers & Verschaffel, 2002; Meira, 1998), müssen Schülerinnen und Schüler für jede Repräsentation, die sie neu kennenlernen, zunächst lernen, wie diese in der Mathematik und in ihrem Unterricht verwendet und interpretiert werden kann. Hinzu kommt, dass Zusammenhänge mit anderen Repräsentationen für dasselbe mathematische Objekt hergestellt werden müssen, um über die spezielle Repräsentation hinauszugehen und in der Lage zu sein, flexibel mit unterschiedlichen Repräsentationen umzugehen (Duval, 2006; Even, 1998; Kaput, 1989).

Folglich spielen vielfältige Repräsentationen eine Doppelrolle beim Lernen von Mathematik: Einerseits sind sie essenziell für den Aufbau eines flexibel einsetzbaren mathematischen Begriffsverständnisses, andererseits können sie das Verstehen von Lernenden durch die damit verbundenen hohen kognitiven Anforderungen aber auch behindern. Damit Lernende von vielfältigen Repräsentationen profitieren können, ist es dementsprechend wichtig, sie dabei zu unterstützen, Zusammenhänge zwischen verschiedenen Repräsentationen zu erkennen und zwischen ihnen zu wechseln.

Spezifisches Noticing, professionelles Wissen und Sichtweisen von angehenden und praktizierenden Lehrkräften zum Umgang mit vielfältigen Repräsentationen im Mathematikunterricht

Vor dem Hintergrund der Bedeutung, die vielfältige Repräsentationen für das Lernen von Mathematik haben, ergeben sich entsprechende Anforderungen an die Lehrkräfte im Mathematikunterricht. Angesichts der Doppelrolle, die vielfältige Repräsentationen spielen, können diese Anforderungen in gewissem Sinne widersprüchlich sein: Einerseits sollten Lernende dazu ermutigt werden, vielfältige Repräsentationen zu nutzen, um sie mit den Voraussetzungen für erfolgreiches mathematisches Denken und Problemlösen auszustatten. Andererseits sollten Schülerinnen und Schüler aber auch vor überfordernden kognitiven Anforderungen bewahrt werden, die häufig mit Repräsentationswechseln verbunden sind und das Lernen behindern können. Diesen Anforderungen gerecht zu werden, kann als ein disziplinspezifisches Dilemma angesehen werden (vgl. Dreher, 2012b; Ball, 1993b), das von Mathematiklehrkräften ausbalanciert werden muss, im Sinne dessen, dass beide Seiten gesehen werden und mit diesen situationsgerecht reflektiert umgegangen wird. Lehrpersonen müssen also in der Unterrichtsinteraktion situationsspezifisch entscheiden, ob die Einführung einer neuen Repräsentation hilfreich ist und Zusammenhänge in ausreichender Weise sichtbar gemacht werden können, oder ob dieser Repräsentationswechsel das Verständnis von Lernenden eher behindert. Insbesondere in Anbetracht dessen, dass Lehrkräfte Repräsentationswechsel häufig nicht bewusst als solche wahrnehmen (z.B. Gerster & Schulz, 2004), ist es folglich bedeutsam, solchen Repräsentationswechseln Beachtung zu schenken und sie vor dem Hintergrund von entsprechendem professionellem Wissen kritisch zu reflektieren.

Versteht man Noticing von Lehrkräften im Sinne der Theorie von van Es & Sherin (2002) als das Bemerken von relevanten Geschehnissen im Mathematikunterricht und das Reflektieren dieser unter Rückgriff auf entsprechende professionelle Wissenskomponenten und Sichtweisen, dann kann das Bemerken und wissensbasierte Reflektieren von Repräsentationswechseln als eine Art spezifisches Noticing angesehen werden. Unter dem Blickwinkel des Umgangs mit vielfältigen Repräsentationen im Mathematikunterricht bezieht sich dieses spezifische Noticing auf fachdidaktische Kriterien im Hinblick auf inhaltlich nicht notwendige und für das Schülerverständnis möglicherweise hinderliche Repräsentationswechsel. Während Wissen und Sichtweisen eine offensichtliche Rolle für die Reflexion des Wahrgenommenen spielen, hängt auch schon das Identifizieren von relevanten Geschehnissen in Unterrichtssituationen von professionellem Wissen und Sichtweisen der Lehrkräfte ab (Bromme, 1992; Kersting, Givvin, Thompson, Santagata & Stigler, 2012; Schwindt, 2008; Schoenfeld, 1998; Sherin, 2007). Schoenfeld (2011) hat in seinem Meta-Artikel über Studien zu „Mathematics Teacher Noticing“ deshalb dazu aufgefordert, Noticing nicht isoliert zu betrachten, sondern Zusammenhänge zwischen Noticing einerseits und Sichtweisen und Wissen von Mathematiklehrkräften andererseits zu untersuchen. Dementsprechend nimmt diese Dissertationsstudie nicht nur spezifisches Noticing von Lehrkräften in den Blick, sondern auch deren professionelles Wissen und Sichtweisen zum Umgang mit vielfältigen Repräsentationen im Mathematikunterricht.

Um solches professionelles Wissen von Lehrkräften zu untersuchen, wird in der vorliegenden Studie das in Abbildung B.1 dargestellte Modell verwendet. Dieses dreidimensionale Modell berücksichtigt das Spektrum zwischen Wissen und Sichtweisen, die Wissensbereiche nach Shulman (1986a) und unterschiedliche Ebenen von Globalität bzw. Situiertheit.

Im Hinblick auf den Umgang mit vielfältigen Repräsentationen im Mathematikunterricht – und insbesondere bezogen auf den Inhaltsbereich Brüche – können beispielsweise verschiedene Komponenten fachdidaktischen Wissens und fachdidaktischer Sichtweisen auf unterschiedlichen Globalitätsebenen bedeutsam sein: Zum einen könnten allgemeine Sichtweisen und Wissensaspekte zur Bedeutung vielfältiger Repräsentationen für das Lernen von Mathematik und insbesondere die Wahrnehmung von Gründen für das Nutzen vielfältiger Repräsentationen im Mathematikunterricht eine Rolle spielen (vgl. Ball, 1993a). Zum anderen sollte aber auch inhaltsbereichsspezifisches professionelles Wissen zum Umgang mit vielfältigen

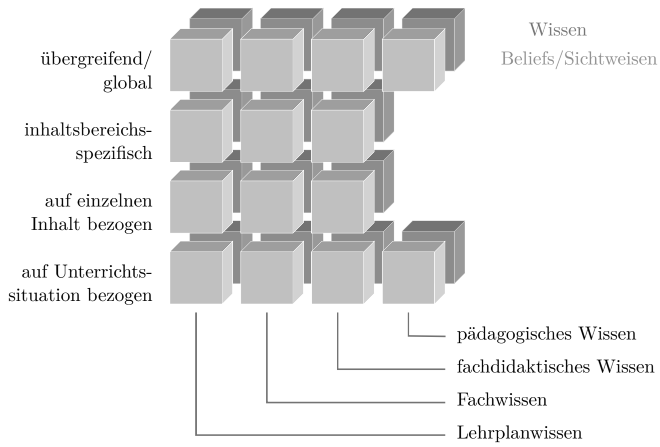


Abbildung B.1: Schematischer Überblick über Komponenten professionellen Wissens (vgl. Kuntze & Zöttl, 2008, S. 48)

Repräsentationen beim Unterrichten des Themas Brüche berücksichtigt werden. Solches Wissen darüber, welche Rolle vielfältige Repräsentationen für den Bruchrechenunterricht spielen, könnte aber weiterhin von professionellem Wissen zu konkreten Inhalten im Bereich Brüche und insbesondere von aufgabenspezifischen Sichtweisen unterschieden werden (vgl. Kuntze, 2012). Im Zusammenhang mit spezifischen Aufgaben zum Bruchrechnen stellt sich beispielsweise die Frage, ob Lehrkräfte das Lernpotential von Aufgaben, die Repräsentationswechsel anregen, erkennen, oder ob sie – wie Ergebnisse einer früheren Studie nahe legen (Kuntze & Dreher, 2014) – den Mehrwert des Nutzens vielfältiger Repräsentationen primär in motivationalen Aspekten sehen, die beispielsweise mit bildlichen Darstellungen in Aufgaben verbunden sein können. Da professionelles Wissen von Lehrkräften häufig an mentale Repräsentationen von Unterrichtssituationen geknüpft ist (Leinhardt & Greeno, 1986), ist des Weiteren auch damit zu rechnen, dass situiertes Wissen, das eng mit spezifischen Unterrichtsepisoden verbunden ist, eine Rolle für den Umgang mit vielfältigen Repräsentationen im Mathematikunterricht spielt.

Diese Unterscheidung von Komponenten fachdidaktischen Wissens zur Rolle vielfältiger Repräsentationen für das Lernen von Mathematik auf verschiedenen Ebenen der Globalität bzw. Situiertheit ermöglichte der Studie zu untersuchen, wie stark solche Komponenten zusammenhängen bzw. ob sie ein konsistentes Bild professionellen Wissens ergeben. Außerdem konnten Antworten auf die Frage gefunden werden, auf welche Komponenten fachdidaktischen Wissens im Spektrum zwischen global und situiert Lehrkräfte beim spezifischen Noticing zurückgegriffen haben. Da damit zu rechnen ist, dass eine Voraussetzung für spezifisches Noticing bezogen auf Repräsentationswechsel darin besteht, dass mathematische Zusammenhänge zwischen verschiedenen Repräsentationen eines mathematischen Objekts inhaltlich verstanden werden, wurde in dieser Studie auch solches spezifisches Fachwissen mit in den Blick genommen.

FORSCHUNGSINTERESSE

Vor dem Hintergrund der oben dargelegten Forschungslücke, angesichts der Bedeutung vielfältiger Repräsentationen für das Lernen von Mathematik auch Lehrkräfte in den Blick zu nehmen, ergibt sich das erste Forschungsinteresse der vorliegenden Dissertationsstudie. Dieses Forschungsinteresse betrifft die Untersuchung von Aspekten professionellen Wissens und von spezifischem Noticing zum Umgang mit vielfältigen Repräsentationen im Mathematikunterricht, insbesondere, um entsprechende Bedarfe an spezifischen Angeboten in der Aus- und Fortbildung von Lehrkräften zu identifizieren.

Das zweite Forschungsinteresse dieser Studie besteht darin, zu einem besseren Verständnis von Zusammenhängen der untersuchten Konstrukte beizutragen. Dies betrifft insbesondere Zusammenhänge zwischen Komponenten spezifischen professionellen Wissens auf verschiedenen Globalitätsebenen und auch Zusammenhänge zwischen spezifischem Noticing und Komponenten professionellen Wissens zum Umgang mit vielfältigen Repräsentationen im Mathematikunterricht. Das dritte Forschungsinteresse dieser Dissertationsstudie bezieht sich auf mögliche Unterschiede zwischen Gruppen von Befragten: Explorativ wurde untersucht, welche Rolle kulturelle Bedingungen in England und Deutschland, verschiedene Professionalisierungsstadien und Schulkulturen auf Aspekte professionellen Wissens, Sichtweisen und auf spezifisches Noticing zum Umgang mit vielfältigen Repräsentationen im Mathematikunterricht spielen.

Forschungsdesign und Methoden

Für die Erhebung von professionellem Wissen, Sichtweisen und spezifischem Noticing von Lehrkräften zum Umgang mit vielfältigen Repräsentationen im Mathematikunterricht wurde ein entsprechendes papierbasiertes Fragebogeninstrument erstellt. Dieser Fragebogen basiert auf einer früheren Version, die im Rahmen einer Pilotstudie mit 145 angehenden Lehrkräften getestet und entsprechend weiterentwickelt wurde. Das auf Deutsch konzipierte Fragebogeninstrument wurde anschließend in eine englische Version übersetzt. Die Übersetzung wurde von zwei englischen Muttersprachlern eingehend geprüft, wobei eine der beiden Personen auch fließend Deutsch spricht und sowohl in England als auch in Deutschland Mathematik unterrichtet hat.

Entsprechend der im theoretischen Hintergrund skizzierten Überlegungen zu verschiedenen potentiell relevanten Aspekten von professionellem Wissen, Sichtweisen und spezifischem Noticing zum Umgang mit vielfältigen Repräsentationen im Mathematikunterricht, umfasst das Fragebogeninstrument mehrere Teile, die auf solche verschiedenen Aspekte abzielen:

- Skalen zur Erhebung von Sichtweisen zu allgemeinen Gründen für das Nutzen vielfältiger Repräsentationen im Mathematikunterricht,
- Skalen zur Erfassung von fachdidaktischen Sichtweisen zum Umgang mit vielfältigen Repräsentationen beim Unterrichten des Inhaltsbereichs Brüche,
- Erfassung aufgabenspezifischer Sichtweisen im Hinblick auf das Lernpotential von zwei Typen von Aufgaben aus dem Inhaltsbereich Brüche, die vielfältige Repräsentationen unterschiedlich nutzen,
- Vignettenbasierter Fragebogenteil zur Erhebung von spezifischem Noticing mit Transkripten von fiktiven Unterrichtssituationen zum Thema Brüche, in denen jeweils ein aus fachdidaktischer Sicht für das Verständnis der Lernenden potentiell hinderlicher Repräsentationswechsel durchgeführt wird, und
- Fachwissenstest zu Repräsentationswechseln im Inhaltsbereich Brüche.

Die meisten Konstrukte im Fokus dieser Studie, insbesondere in den Fragebogenteilen zu Sichtweisen von Lehrkräften, wurden mit Hilfe von Likert-Skalen erfasst. Um die theoriebasierte Struktur dieser Fragebogeninstrumente empirisch zu überprüfen, wurden konfirmatorische Faktorenanalysen durchgeführt. Unterschiede zwischen zwei Teilstichproben entsprechend des dritten Forschungsinteresses wurden mittels T-Tests untersucht. Zusammenhänge zwischen unterschiedlichen Aspekten professionellen Wissens, Sichtweisen und Noticing wurden auf quantitative Weise durch die Berechnung von Korrelationskoeffizienten nach Pearson ermittelt. Ergänzende qualitative Auswertungen betreffen nicht nur die top-down Doppelkodierung von Antworten von Teilnehmerinnen und Teilnehmern in Bezug auf den vignetten-basierten Fragebogenteil, sondern umfassen auch die Analyse von Fällen im Sinne einer Konsenskodierung im Hinblick auf die Frage, auf welche Komponenten professionellen Wissens einzelne Lehrkräfte bei ihrem spezifischen Noticing zurückgegriffen haben.

Im Hinblick auf das dritte Forschungsinteresse dieser Dissertationsstudie wurden unterschiedliche Gruppen von angehenden und praktizierenden Lehrpersonen untersucht:

- 139 angehende Lehrkräfte im Primarbereich in England,
- 219 angehende Lehrkräfte im Primarbereich in Deutschland,
- 67 angehende Lehrkräfte für das Gymnasium (in Deutschland),
- 77 praktizierende Lehrkräfte am Gymnasium (in Deutschland) und
- 25 praktizierende Lehrkräfte an der Haupt-/Werkrealschule (in Deutschland).

Die angehenden und praktizierenden Lehrkräfte beantworteten den Fragebogen unter Aufsicht der Autorin oder einer Hilfskraft.

Um Antworten auf die Forschungsfragen der Studie zu finden, wurden drei Teilstudien betrachtet, zu denen entsprechende Artikel verfasst wurden. Jede dieser Teilstudien nimmt einen anderen möglichen Einflussfaktor entsprechend des dritten Forschungsinteresses in den Blick und vergleicht dementsprechend zwei Gruppen von Befragten: Die Teilnehmenden der ersten Teilstudie sind angehende Lehrkräfte in England und Deutschland, die zweite Teilstudie befasst sich mit dem Vergleich von angehenden und praktizierenden Lehrkräften und die dritte Teilstudie fokussiert praktizierende Lehrkräfte an unterschiedlichen Sekundarschultypen (Gymnasium bzw. Haupt-/Werkrealschule). Diese drei Teilstudien unterscheiden sich jedoch nicht nur in den Teilstichproben, die betrachtet wurden, sondern sie beschäftigen sich zudem auch mit unterschiedlichen Aspekten bezüglich der ersten beiden Forschungsinteressen der vorliegenden Dissertationsstudie und ergänzen sich somit gegenseitig. Während die erste Teilstudie sich primär auf Sichtweisen von angehenden Lehrkräften zum Umgang mit vielfältigen Repräsentationen auf unterschiedlichen Globalitätsebenen und auf entsprechende Zusammenhänge konzentrierte, wurden in der zweiten Teilstudie zum einen zusätzlich spezifisches Noticing und Zusammenhänge mit verschiedenen Komponenten professionellen Wissens untersucht und zum anderen wurden auch praktizierende Lehrkräfte mit in den Blick genommen. Die dritte Teilstudie untersuchte Aspekte professionellen Wissens, Sichtweisen und spezifisches Noticing ebenso wie entsprechende Zusammenhänge noch einmal speziell unter dem Blickwinkel des Dilemmas von vielfältigen Repräsentation zwischen Lernhilfe und Lernhürde.

Ergebnisse, Diskussion und Folgerungen

Entsprechend der drei Forschungsinteressen dieser Dissertationsstudie haben ihre Ergebnisse dreierlei Funktionen: Erstens können sie Einblick gewähren in Wissenskomponenten, Sichtweisen und spezifisches Noticing von angehenden und praktizierenden Lehrkräften in Bezug auf die Rolle vielfältiger Repräsentationen beim Lernen von Mathematik. Dies erlaubt insbesondere die Feststellung spezifischer Voraussetzungen und Bedarfe für die Aus- und Fortbildung von Mathematiklehrkräften. Zweitens können die Befunde zu einem besseren Verständnis von Zusammenhängen zwischen unterschiedlichen Aspekten von professionellem Wissen, Sichtweisen und Noticing betragen. Und drittens erlaubt der Vergleich unterschiedlicher Teilstichproben das Identifizieren von Unterschieden in Zusammenhang mit verschiedenen Professionalisierungsstadien, unterschiedlichen kulturellen Bedingungen und verschiedenen Schulkulturen.

In Bezug auf Voraussetzungen (deutscher) Lehramtsstudierenden weisen die Ergebnisse zu allgemeinen, inhaltsbereichsspezifischen sowie aufgabenspezifischen Sichtweisen zum Nutzen vielfältiger Repräsentationen im Mathematikunterricht auf eine geringe Sensibilisierung bezüglich der Schlüsselrolle vielfältiger Repräsentationen für das Lernen von Mathematik hin. Hinzu kommt ein im Durchschnitt relativ geringes spezifisches Fachwissen, das anscheinend teilweise auch eine Hürde für spezifisches Noticing war. Hinweise auf einen solchen Zusammenhang geben sowohl eine signifikant positive Korrelation zwischen diesen beiden Konstrukten als auch Antworten einzelner Lehramtsstudierender, die zeigen, dass die gegebenen Repräsentationen inhaltlich nicht verstanden wurden, was zu einer nicht adäquaten Beurteilung beitrug. Die potentiell hinderliche Rolle von inhaltlich nicht notwendigen Repräsentationswechseln in spezifischen Unterrichtssituationen wurde von den angehenden Lehrkräften relativ selten erkannt. Die vorliegenden Befunde zeigen einen Bedarf der untersuchten Lehramtsstudierenden an spezifischen Lerngelegenheiten im Rahmen ihrer Lehramtsausbildung auf. Solche Lernumgebungen für angehende Lehrkräfte sollten insbesondere situierte Aspekte des Umgangs mit vielfältigen Repräsentationen integrieren, beispielsweise indem mit Unterrichtsvideos und Unterrichtsmaterial gearbeitet wird, das spezifische Inhalte fokussiert (vgl. Dreher & Kuntze, 2012). Neben der Förderung spezifischen Fachwissens sollte ein wichtiger Bestandteil derart situierter Ansätze der Lehramtsausbildung die Verbindung mit theoretischen Aspekten der Rolle vielfältiger Repräsentationen für das Lernen von Mathematik sein.

Ergebnisse im Hinblick auf die praktizierenden Lehrkräfte, die in dieser Studie untersucht wurden, deuten darauf hin, dass sie sich von den angehenden Lehrkräften insbesondere in Bezug auf eher situierte Aspekte des Umgangs mit vielfältigen Repräsentationen und vor allem hinsichtlich ihres spezifischen Noticings unterscheiden. Dennoch zeigen die Befunde, dass auch erfahrene Mathematiklehrkräfte nicht notwendig Experten im Umgang mit vielfältigen Repräsentationen sind. Besonders auf der Ebene von globalen Sichtweisen messen sie disziplinspezifischen Gründen für das Nutzen vielfältiger Repräsentationen im Mathematikunterricht durchschnittlich verhältnismäßig wenig Bedeutung bei. So wurde der Mehrwert vielfältiger Darstellungen eher darin gesehen, dass das Merken von Fakten unterstützt und das Interesse von Lernenden gesteigert werden könne, als in der Bedeutung von Repräsentationswechseln für mathematisches Verständnis. Spezifisch zugeschnittene Fortbildungsangebote sollten demensprechend insbesondere ein Augenmerk richten auf die disziplinspezifische Rolle vielfältiger Repräsentationen für ein flexibel einsetzbares Begriffsverständnis, im Sinne einer übergreifenden Idee, die theoriebasiertes Planen und Reflektieren von Unterrichtsprozessen unterstützen kann (vgl. Kuntze, Lerman, Murphy, Kurz-Milcke, Siller & Winbourne, 2011). Solche Fortbildungsangebote könnten abzielen auf die Verknüpfung der eigenen Unterrichtserfahrungen der Lehrkräfte mit einer theoretischen Perspektive zur Rolle vielfältiger Repräsentationen. Diese theoretische Perspektive könnte dann wiederrum dem spezifischen Noticing bezüglich des eigenen Unterrichts dienen.

In Bezug auf Zusammenhänge zwischen Komponenten professionellen Wissens und entsprechender Sichtweisen der Befragten auf verschiedenen Globalitätsebenen, legen die Ergebnisse dieser Studie nahe, dass kein einfacher Transfer zwischen globalem und situierterem professionellem Wissen angenommen werden kann. Folglich stützen diese Befunde die Annahme, dass solche Komponenten professionellen Wissens auf verschiedenen Globalitätsebenen als eigene Konstrukte betrachtet werden können (vgl. Kuntze, 2012).

Hinsichtlich des Zusammenspiels von spezifischem Noticing und Komponenten professionellen Wisses, können die Ergebnisse dieser Studie insofern zu einem besseren Verständnis beitragen, als sich gezeigt hat, dass Lehrkräfte bei spezifischem Noticing auf ganz unterschiedliche Komponenten ihres professionellen Wissens zurückgegriffen haben. Insbesondere konnten solche Komponenten auf unterschiedlichen Globalitätsebenen lokalisiert werden.

Die Vergleiche zwischen Gruppen von Befragten ergaben nicht nur Unterschiede zwischen angehenden und praktizierenden Lehrkräften: Sie zeigen außerdem, dass kulturspezifische Aspekte von Sichtweisen zur Rolle vielfältiger Repräsentationen berücksichtigt werden sollten. Des Weiteren weisen sie auch auf einen besonderen Bedarf von Lehrkräften an Haupt-/Werkrealschulen an speziellen Fortbildungsangeboten hin, da diese Lehrkräfte im Durchschnitt eine geringere Sensibilisierung für die Doppelrolle vielfältiger Repräsentationen beim Lernen von Mathematik gezeigt haben als ihre Kolleginnen und Kollegen an Gymnasien.

Die Ergebnisse dieser Dissertationsstudie können als Ausgangspunkt für zukünftige Forschungsprojekte dienen: Derartige Studien sollten zum einen eine breitere Perspektive einnehmen, indem auch andere Inhaltsbereiche in den Blick genommen werden, aber zum anderen auch bestimmte Aspekte weitergehend untersuchen. So könnte spezifisches Noticing beispielsweise zusätzlich mit Hilfe von Videovignetten erhoben werden, um ein realistischeres Bild von Unterrichtssituationen zu zeichnen.

Anhang C

Darlegung des eigenen Anteils

Der theoretische Hintergrund dieser Dissertationsstudie wurde von Anika Dreher in weiten Teilen selbst erarbeitet. Dazu gehört nicht nur die Recherche und Verknüpfung bereits existierender Theorien, sondern auch die Entwicklung eigener Vermutungen und Ideen. Vor diesem Hintergrund und unter Berücksichtigung weiterer Ideen und Anregungen von Prof. Dr. Sebastian Kuntze wurden die Forschungsfragen dieses Dissertationsprojekts abgesteckt. Um Antworten auf diese Forschungsfragen zu finden, hat Anika Dreher in Zusammenarbeit mit Prof. Dr. Sebastian Kuntze ein eigens zugeschnittenes Fragebogeninstrument entwickelt. Der Anteil von Prof. Dr. Sebastian Kuntze bezieht sich dabei vor allem auf Ideen bezüglich des Designs von Fragebogenteilen, insbesondere im Hinblick auf den aufgabenbezogenen Teil, auf Vorschläge für Multiple-Choice Items und auf Formulierungsänderungen. Die Vignetten zu Unterrichtssituationen, die Aufgaben zu Brüchen und die Items des Fachwissenstest wurden von Anika Dreher jedoch größtenteils selbst entwickelt.

Für die Rekrutierung der Lehrpersonen und Lehramtsstudierenden, sowie für Organisation und Durchführung der Studie war Anika Dreher zuständig. Die Aufsicht von teilnehmenden Personen während der Bearbeitung des Fragebogeninstruments wurde teilweise von Hilfskräften unterstützt.

Die Daten der Studie wurden von Anika Dreher eigenständig ausgewertet, wobei sie von Prof. Dr. Sebastian Kuntze durch gemeinsame Diskussion der Ergebnisse und Hinweise auf weitere Analysemöglichkeiten beratend unterstützt wurde. Für die Kodierung der Antworten zu den offenen Frageformaten wurde eine Hilfskraft von Anika Dreher geschult, so dass anschießend alle Antworten von diesen beiden Personen unabhängig voneinander doppelkodiert werden konnten.

Die in dieser Dissertationsarbeit abgedruckten wissenschaftlichen Artikel sowie deren Rahmung wurden verfasst von Anika Dreher unter Berücksichtigung vieler wertvoller Anregungen und einiger Formulierungsänderungen von Prof. Dr. Sebastian Kuntze und Prof. em. Dr. Stephen Lerman.

Die Phase der Datenerhebung dieser Dissertationsstudie wurde gefördert im Rahmen des Projekts ABCmaths, das mit Unterstützung der Europäischen Kommission (503215-LLP-1-2009-1-DE-COMENIUS-CMP) finanziert wurde. Diese Veröffentlichung gibt lediglich die Sichtweisen der Autoren wieder, die Kommission haftet nicht für die weitere Verwendung der darin enthaltenen Angaben.

Des Weitern steht die Studie in engem Zusammenhang mit der Arbeit im Projekt La viDa-M („Lernen anregen mit vielfältigen Darstellungen im Mathematikunterricht“), das durch Forschungsmittel des Senats der PH Ludwigsburg gefördert wurde.

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